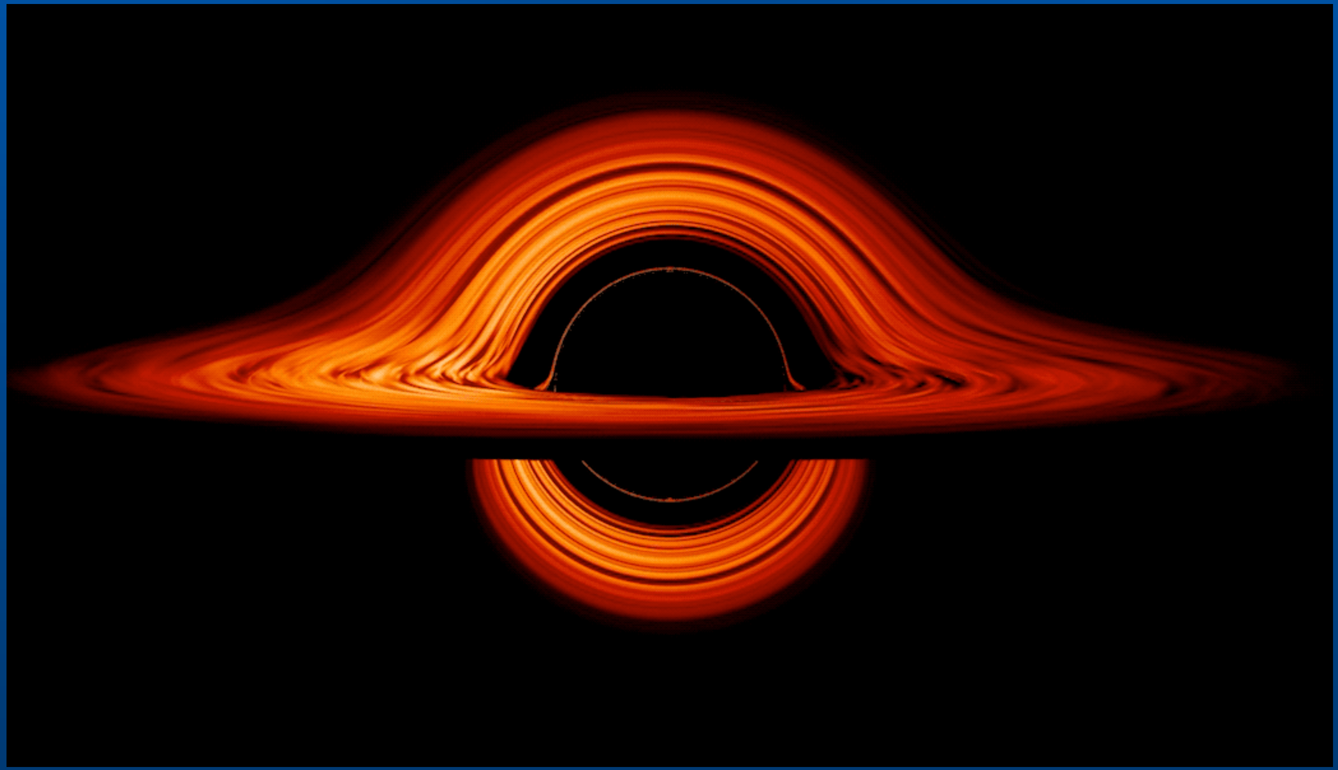
GARRA Group

Black hole astrophysics



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Curso 2022

Some books on General Relativity

- *Introducing Einstein's Relativity*, by R. D'Inverno, Clarendon Press, Oxford, 1992.
- *Space-Time and Geometry*, by S. Carroll, Addison Wesley, San Francisco, 2004.
- *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity,* by Steven Weinberg, Wiley & Sons, NY, 1972.
- *General Relativity*, by M. P. Hobson, G. Efstathiou and A. N. Lasenby, Cambridge University Press, Cambridge, 2006.
- *General Relativity and the Einstein's Equations*, by Y. Choquet-Bruhat, Oxford University Press, Oxford, 2009.*
- *General Relativity*, by R. M. Wald, The University of Chicago Press, Chicago, 1984.*
- *Relativity on Curved Manifolds*, by F. de Felice & C.J.S. Clarke, Cambridge University Press, Cambridge, 1990.*

Some books on General Relativity

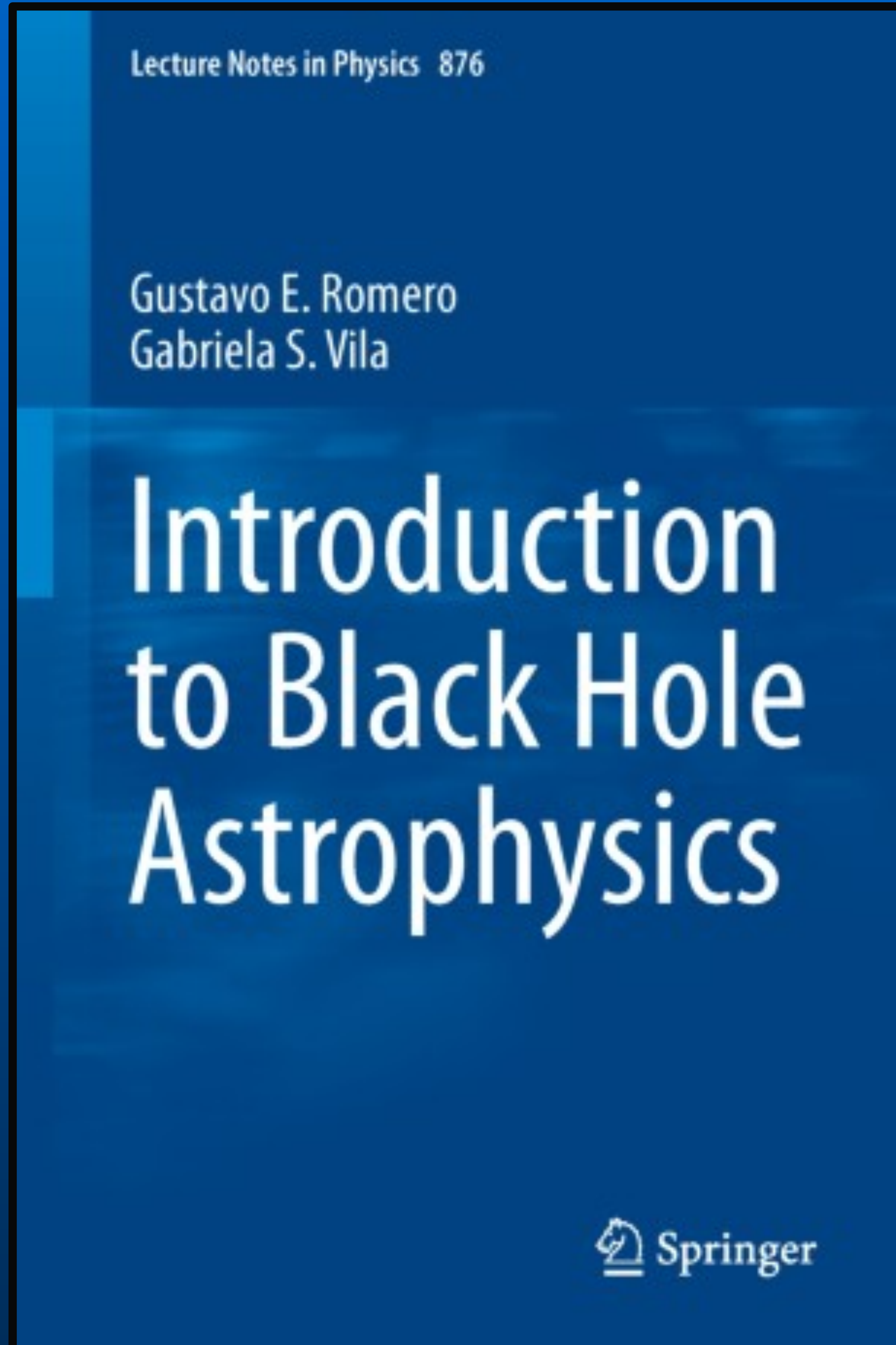
- *A First Course in General Relativity*, by B. Schutz, Cambridge University Press, Cambridge, 2009.
- *Gravitation*, by C. Misner, K. Thorne and J.A. Wheeler, Freeman, San Francisco, 1973.
- *A Short Course in General Relativity*, by J. Foster and J. D. Nightingale, Springer, NY, 2006.
- *Relativity*, by W. Rindler, Oxford University Press, Oxford, 2006.
- *The Large Scale Structure of Space Time*, by S. Hawking and G.F.R. Ellis, Cambridge University Press, Cambridge, 1973.*
- *Global Aspects in Gravitation and Cosmology*, by P.S. Joshi, Oxford University Press, Oxford, 1993.*

Books on black holes

- *Black Holes*, by D. Raine, E. Thomas, Imperial College Press, London, 2005.
- *Introduction to Black Hole Physics*, by V.P. Frolov and A. Zelnikov, Oxford University Press, Oxford, 2011.
- *Black Holes, White Dwarfs and Neutron Stars*, by S. L. Shapiro and S. A. Teukolsky, John Wiley & Sons, New York, 1983.
- *Black Hole Physics*, by V. P. Frolov and I. D. Novikov, Kluwer Academic Publishers, Dordretch, 1998.*

Books on black hole astrophysics

- Beskin V. S., *MHD Flows in Compact Astrophysical Objects* 1st Edition, Springer (2010).
- Camenzind M., *Compact Objects in Astrophysics - White Dwarfs, Neutron Stars and Black Holes,* Springer (2007).
- Longair M., *High-Energy Astrophysics,* Cambridge University Press, Cambridge (2004).
- Frank J., King A. & Raine D., *Accretion Power in Astrophysics,* Cambridge University Press, Cambridge (2002)


Lecture Notes in Physics 876
Gustavo E. Romero
Gabriela S. Vila
Introduction to Black Hole Astrophysics
Springer

¿What is a black hole?

A black hole is an object that has gravitationally collapsed in a full manner.

The properties of a black hole then depend on what is gravity and how it behaves.

Different theories of gravity will yield different black hole models

Newtonian gravity → Black stars

General relativity → Black holes

Modified relativistic gravity → Modified black holes

Newtonian gravity

$$\nabla^2\Phi(\vec{r}) = 4\pi G\rho(\vec{r}),$$

Poisson equation

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}.$$

Laplacian

$$\vec{g} = -\vec{\nabla}\Phi.$$

Force

$$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho(\vec{r}).$$

$$\oint_{\partial V}\vec{g}\cdot d\vec{A} = -4\pi G\int_V \rho(\vec{r})dV$$

Integrating and applying Gauss theorem

$$\oint_{\partial V}\vec{g}(\vec{r})\cdot d\vec{A} = -4\pi GM.$$

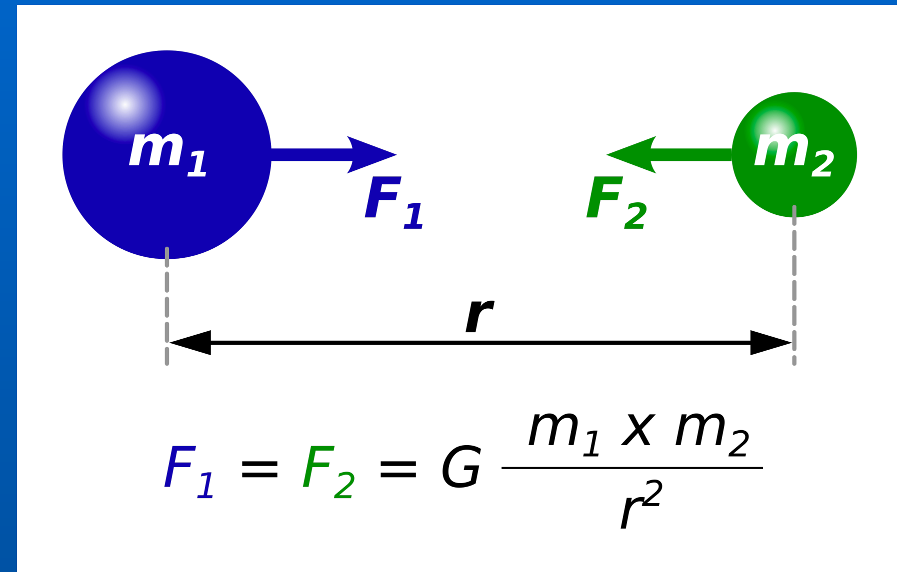
On Earth surface: g = 9.81 m/s^2.

With spherical symmetry:

$$-4\pi GM = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta r^2 g(r) = 4\pi r^2 g(r)$$

$$\boxed{g(r) = -\frac{GM}{r^2}}$$

Newton’s law for two masses



The work needed to move the body over a small distance *dr* against this force is therefore given by

$$dW = F\,dr = G\frac{Mm}{r^2}\,dr.$$

The total work needed to move the body from the surface r_0 of the gravitating body to infinity is then

$$W = \int_{r_0}^{\infty} G\frac{Mm}{r^2}\,dr = G\frac{Mm}{r_0} = mgr_0.$$

$$\frac{1}{2}mv_0^2 = G\frac{Mm}{r_0},$$

which results in

$$v_0 = \sqrt{\frac{2GM}{r_0}} = \sqrt{2gr_0}.$$

Escape velocity

For Earth, the average escape velocity from sea level is 11.19 km/s (kilometers per second), which is equivalent to 40,280 km/h (kilometers per hour).

Tabla de velocidades de escape

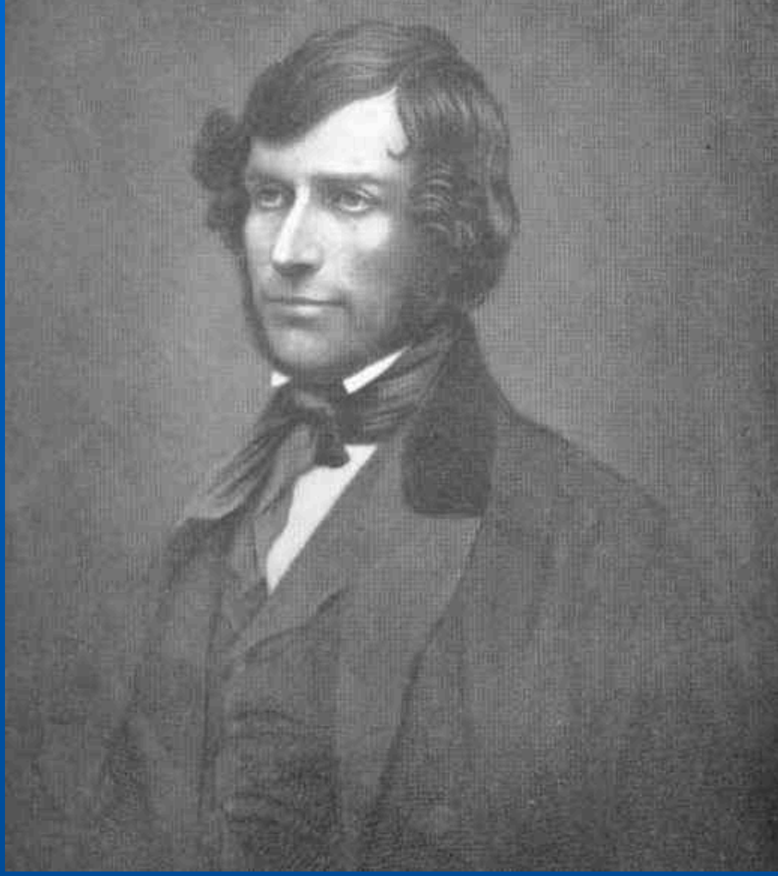
Objeto	Masa (kg)	Radio (m)	Velocidad de escape[1] (km/s)	con respecto a la Tierra
Sol	$2{,}0 \times 10^{30}$	$7{,}0 \times 10^{8}$	617,5	55,18
Mercurio	$3{,}3 \times 10^{23}$	$2{,}4 \times 10^{6}$	4,3	0,38
Venus	$4{,}9 \times 10^{24}$	$6{,}1 \times 10^{6}$	10,4	0,92
Tierra	$6{,}0 \times 10^{24}$	$6{,}4 \times 10^{6}$	11,2	1
Luna	$7{,}3 \times 10^{22}$	$1{,}7 \times 10^{6}$	2,38	0,21
Marte	$6{,}4 \times 10^{23}$	$3{,}4 \times 10^{6}$	5	0,45
Ceres	$9{,}4 \times 10^{20}$	$4{,}9 \times 10^{5}$	0,5	0,04
Júpiter	$1{,}9 \times 10^{27}$	$7{,}1 \times 10^{7}$	59,5	5,32

$\frac{M}{r}$: compactness

Black stars

P.S. Laplace

J. Michell

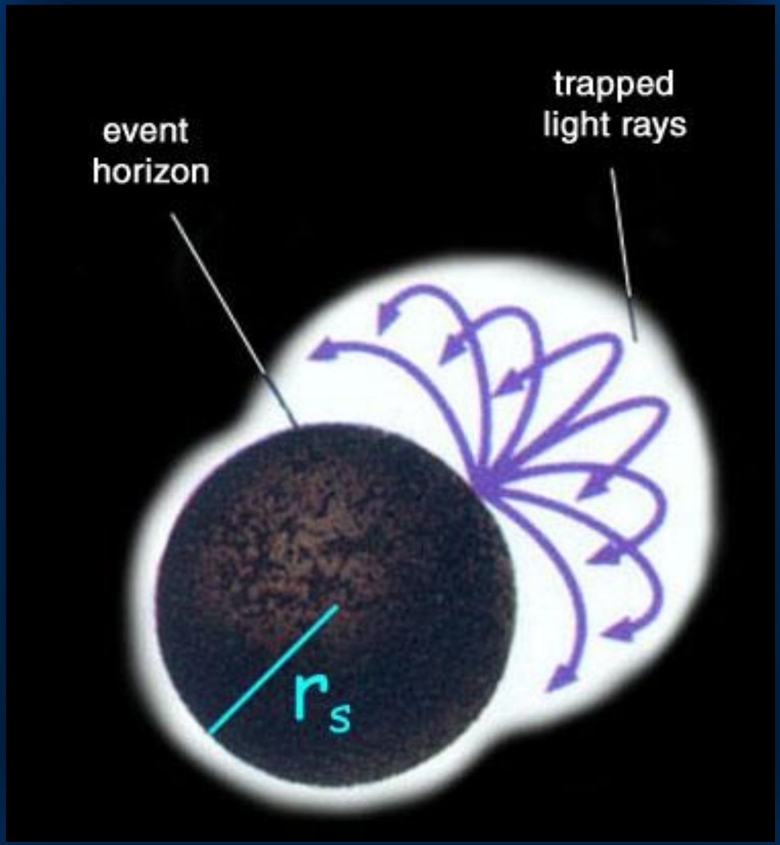


$$v_0 = \sqrt{\frac{2GM}{r_0}}$$

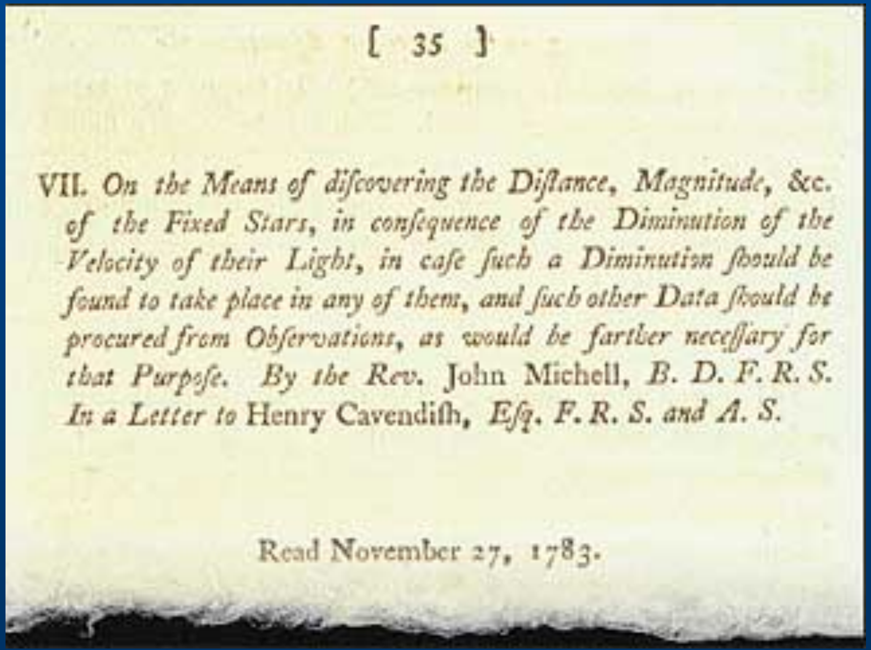
Exposition du système du monde, 1796

Philosophical Transactions of the Royal Society. 74: 35–57., 1784

If $v_0 = c$, light is trapped within a body of size r_s



$$r_s = \frac{2GM}{c^2}$$

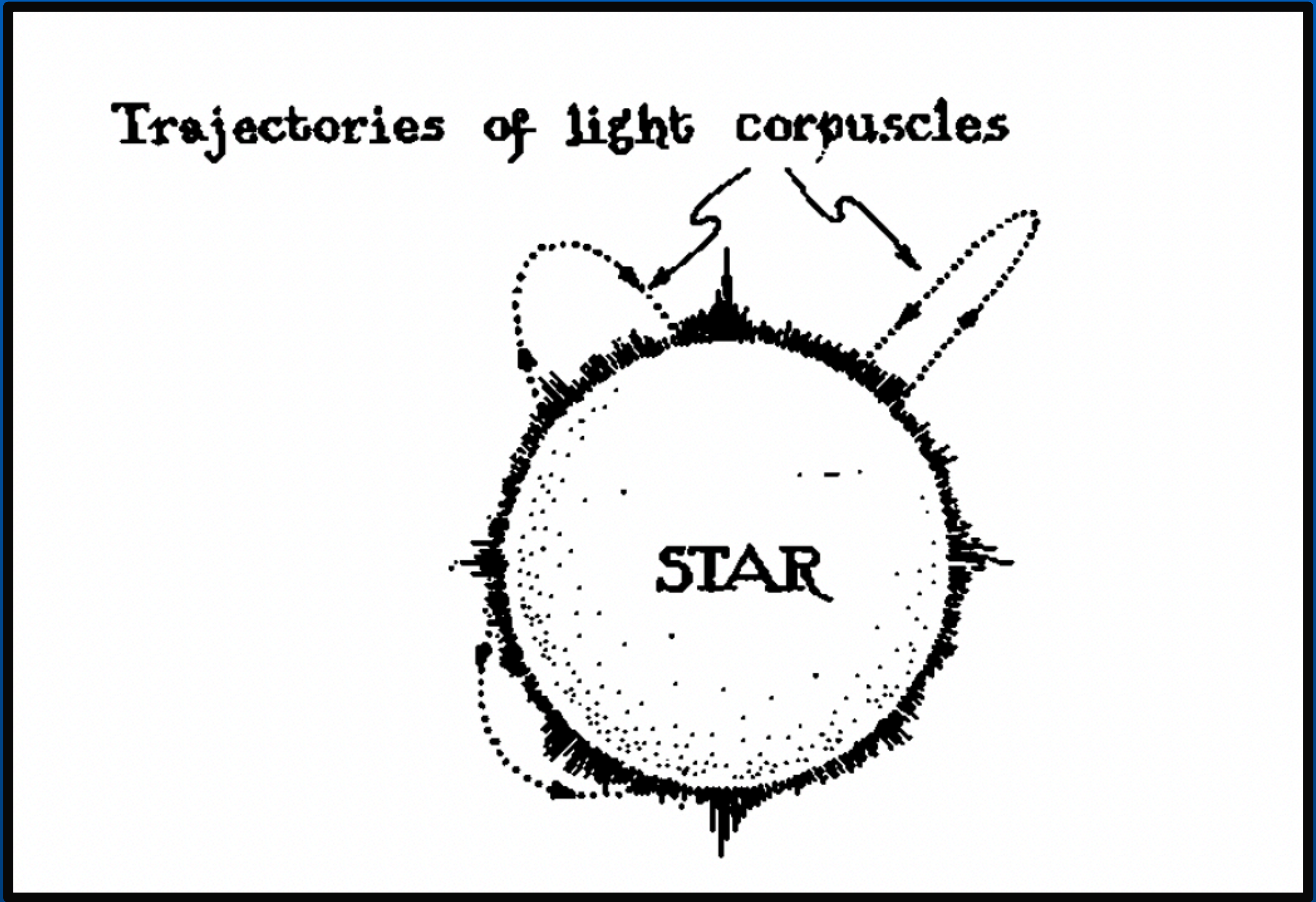


[35]

VII. *On the Means of discovering the Distance, Magnitude, &c. of the Fixed Stars, in consequence of the Diminution of the Velocity of their Light, in case such a Diminution should be found to take place in any of them, and such other Data should be procured from Observations, as would be farther necessary for that Purpose. By the Rev.* John Michell, *B. D. F. R. S. In a Letter to* Henry Cavendish, *Esq. F. R. S. and A. S.*

Read November 27, 1783.

"On the Means of Discovering the Distance, Magnitude, &c. of the Fixed Stars, in Consequence of the Diminution of the Velocity of Their Light, in Case Such a Diminution Should be Found to Take Place in any of Them, and Such Other Data Should be Procured from Observations, as Would be Farther Necessary for That Purpose. By the Rev. John Michell, B. D. F. R. S. In a Letter to Henry Cavendish, Esq. F. R. S. and A. S."



Black stars are not black holes, because they have a solid surface.

The Poisson equation for the gravitational potential is not Lorentz invariant, and hence incompatible with the current view of physics.

James Clerk Maxwell (1831-1879): electromagnetism

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

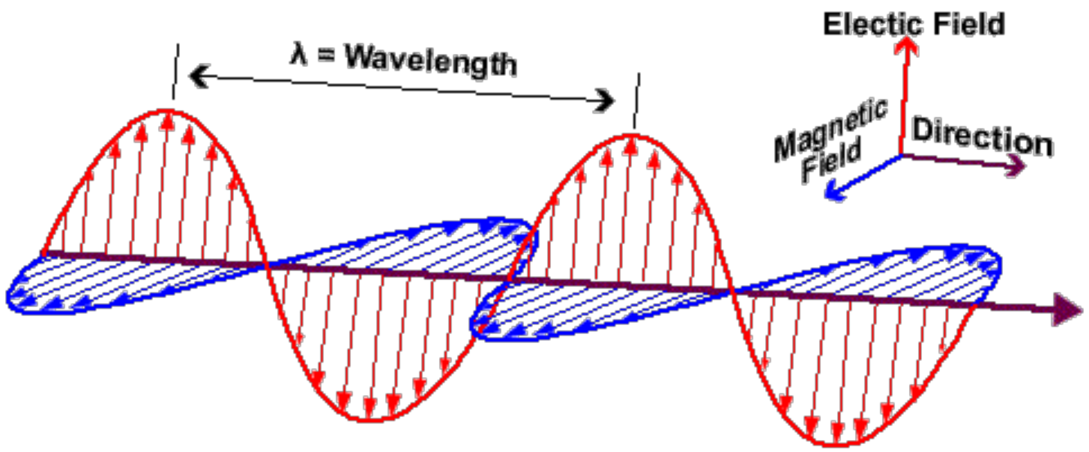
$$\mathbf{F} = q\left[\mathbf{E} + (\mathbf{v} \times \mathbf{B})\right]$$

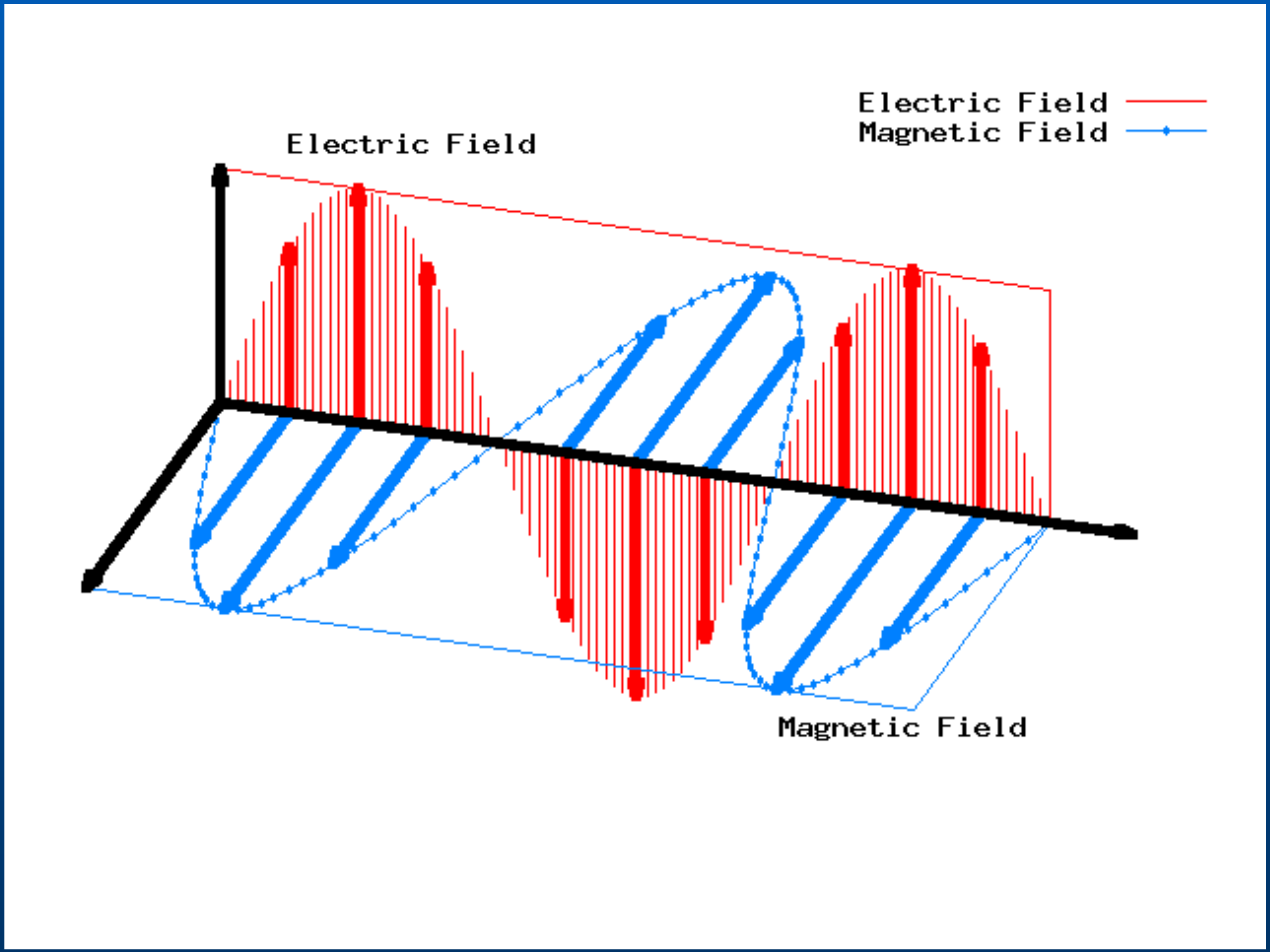
Oliver Heaviside (1850 – 1925)

Ondas electromagnéticas

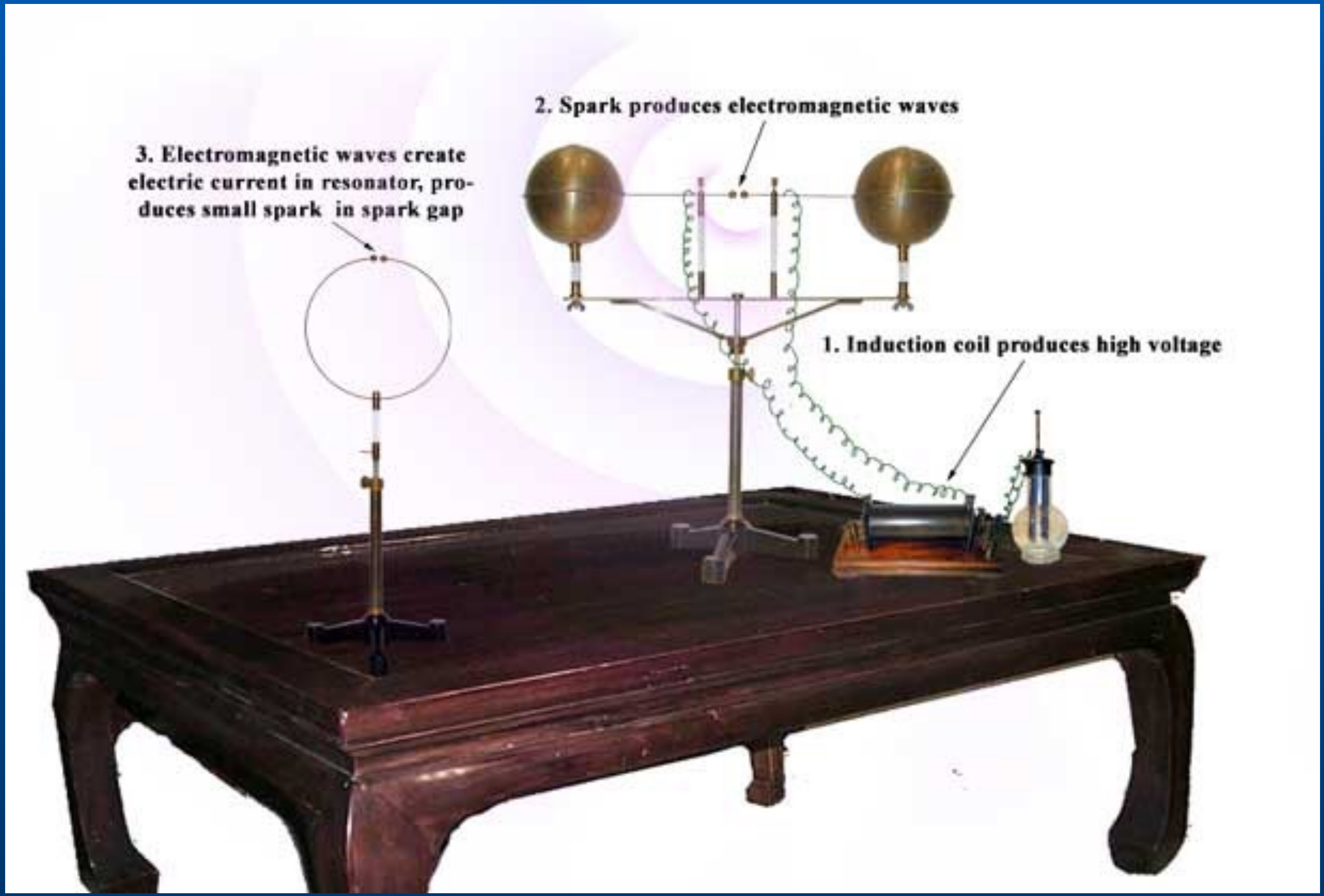
$$\left(c^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\mathbf{E} = \mathbf{0}$$

$$\left(c^2\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \mathbf{0}$$




Electric Field
Magnetic Field
Electric Field
Magnetic Field

Heinrich Hertz 1857-1894



1887

Tension between classical mechanics and electrodynamics

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

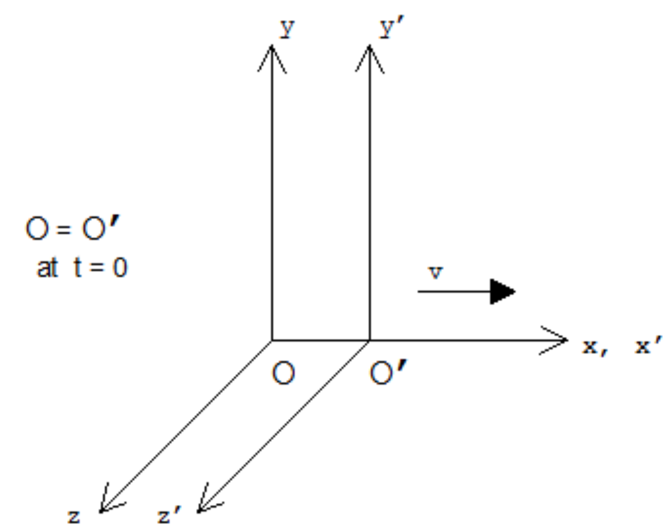
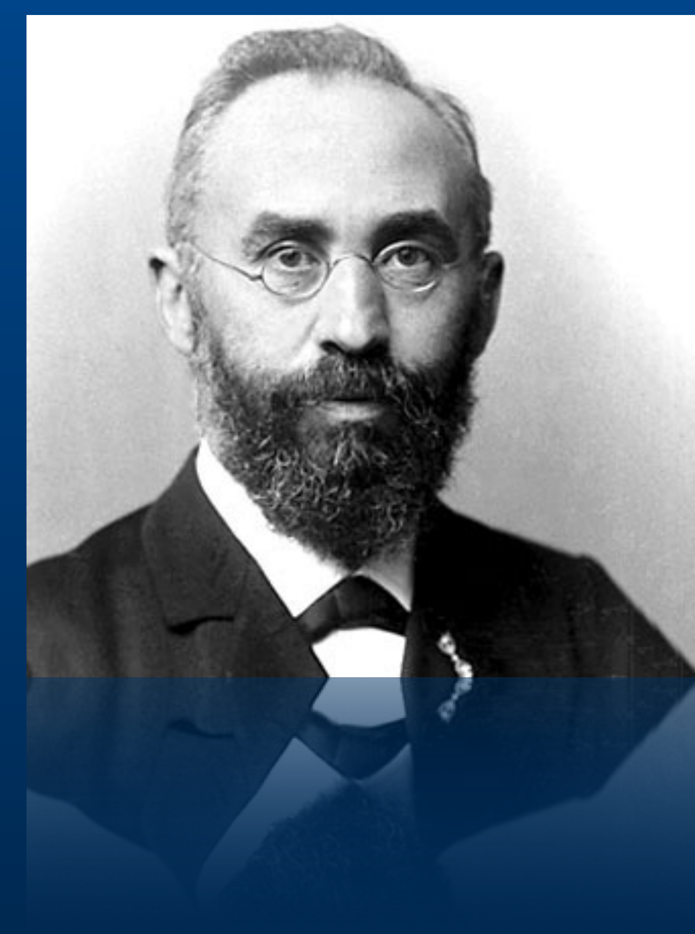


Figure 1. Two Inertial Frames in Standard Configuration

The Lorentz Transformation

(for motion in the x-direction)

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t' = \frac{t - (v/c^2)\,x}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Hendrik Lorentz (1853-1928)

Ann.Physik **17** (1905), 891-921.

Special Relativity

Albert Einstein

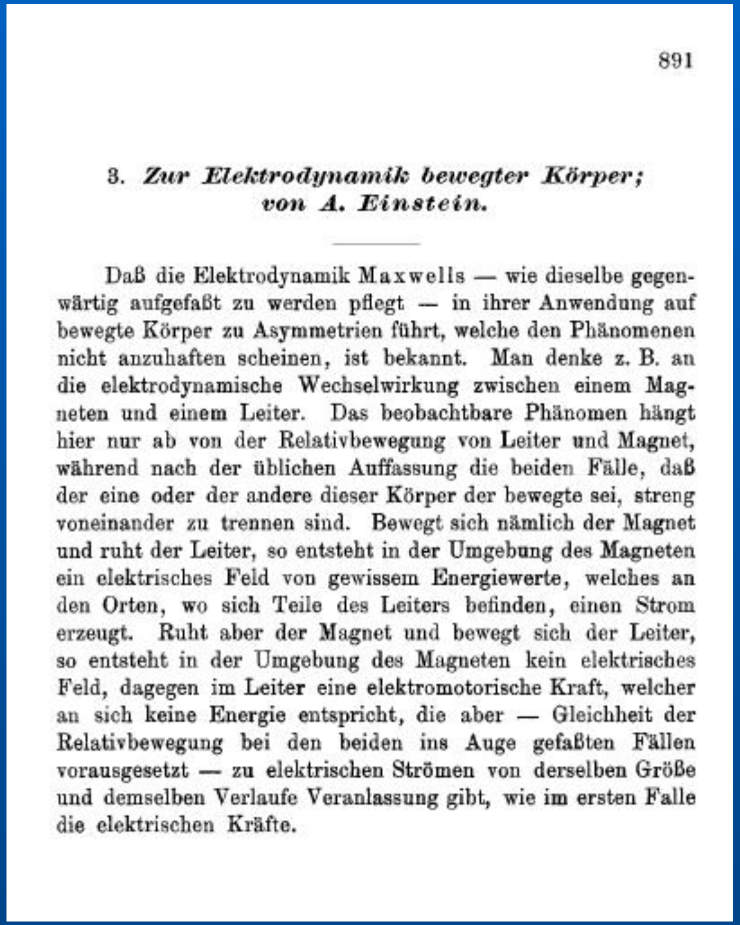
$$p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m_0 v$$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}.$$

$$\begin{aligned} \text{total energy E} &= \text{rest energy } + \text{KE} \\ &= (mc^2) \quad + \quad (\gamma - 1)\, mc^2 \\ &= \gamma mc^2 \end{aligned}$$

$$KE = \frac{1}{2} m_0 v^2 + \frac{3}{8}\frac{m_0 v^4}{c^2} + \frac{5}{16}\frac{m_0 v^6}{c^4} + \ldots$$

$$KE \approx \frac{1}{2} m_0 v^2 \quad for \quad v << c$$



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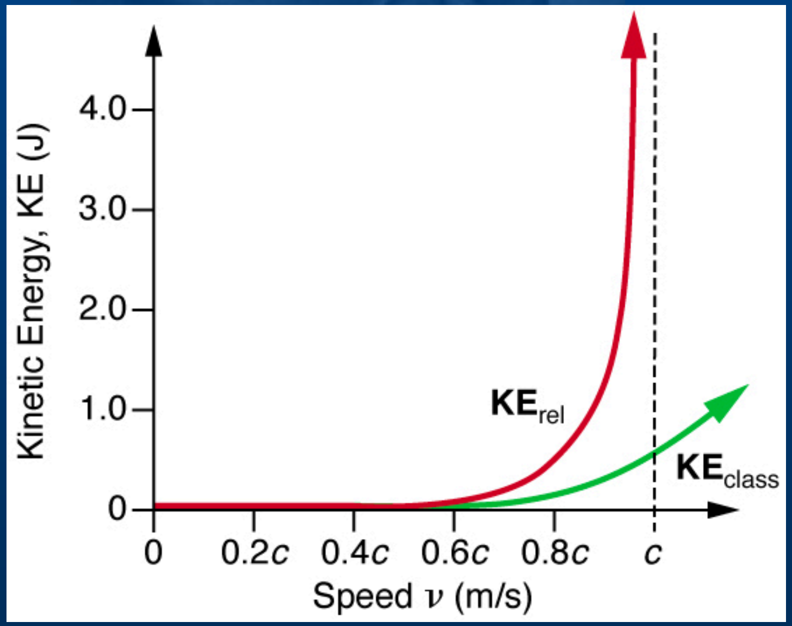
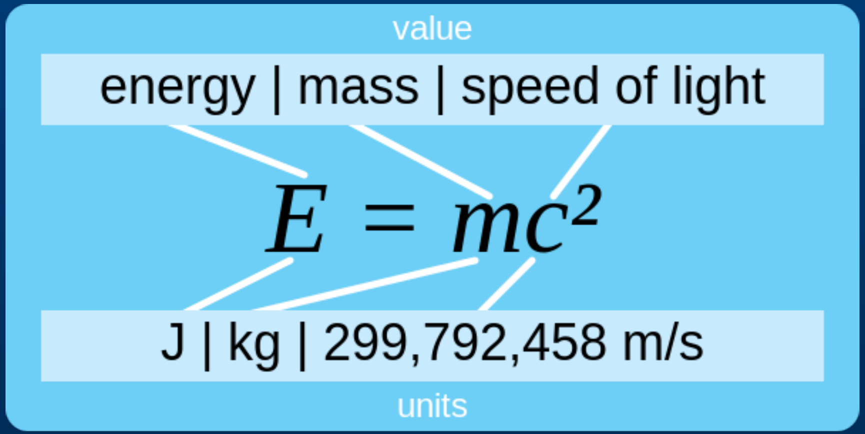
3. *Zur Elektrodynamik bewegter Körper;* *von A. Einstein.*

Daß die Elektrodynamik Maxwells — wie dieselbe gegenwärtig aufgefaßt zu werden pflegt — in ihrer Anwendung auf bewegte Körper zu Asymmetrien führt, welche den Phänomenen nicht anzuhaften scheinen, ist bekannt. Man denke z. B. an die elektrodynamische Wechselwirkung zwischen einem Magneten und einem Leiter. Das beobachtbare Phänomen hängt hier nur ab von der Relativbewegung von Leiter und Magnet, während nach der üblichen Auffassung die beiden Fälle, daß der eine oder der andere dieser Körper der bewegte sei, streng voneinander zu trennen sind. Bewegt sich nämlich der Magnet und ruht der Leiter, so entsteht in der Umgebung des Magneten ein elektrisches Feld von gewissem Energiewerte, welches an den Orten, wo sich Teile des Leiters befinden, einen Strom erzeugt. Ruht aber der Magnet und bewegt sich der Leiter, so entsteht in der Umgebung des Magneten kein elektrisches Feld, dagegen im Leiter eine elektromotorische Kraft, welcher an sich keine Energie entspricht, die aber — Gleichheit der Relativbewegung bei den beiden ins Auge gefaßten Fällen vorausgesetzt — zu elektrischen Strömen von derselben Größe und demselben Verlaufe Veranlassung gibt, wie im ersten Falle die elektrischen Kräfte.

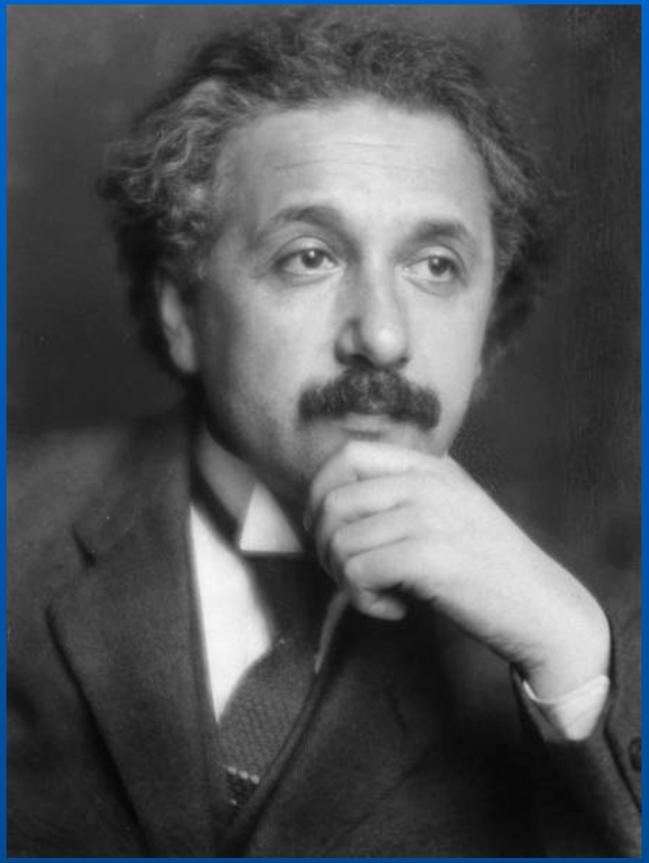
Electrodynamics of moving bodies





$c^2 = 89{,}875{,}517{,}873{,}681{,}800\ m^2/s^2$

The road to General Relativity

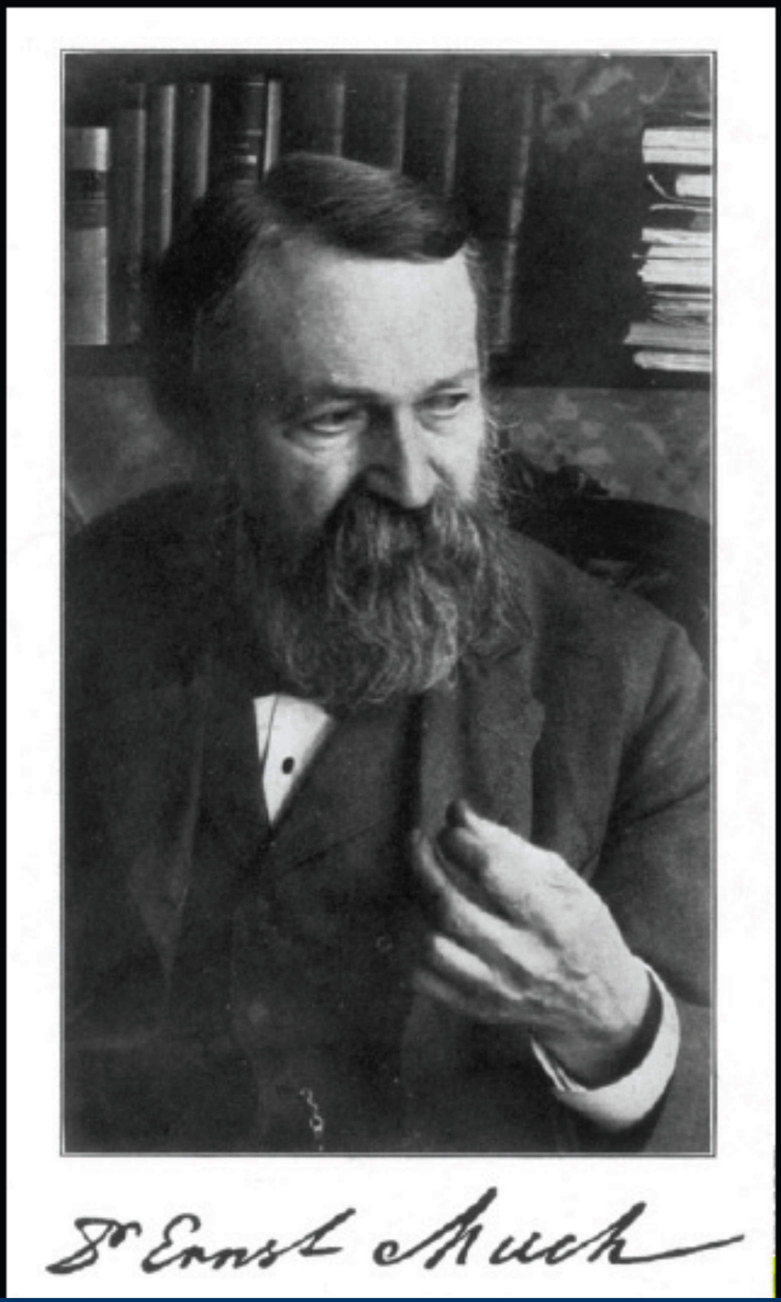


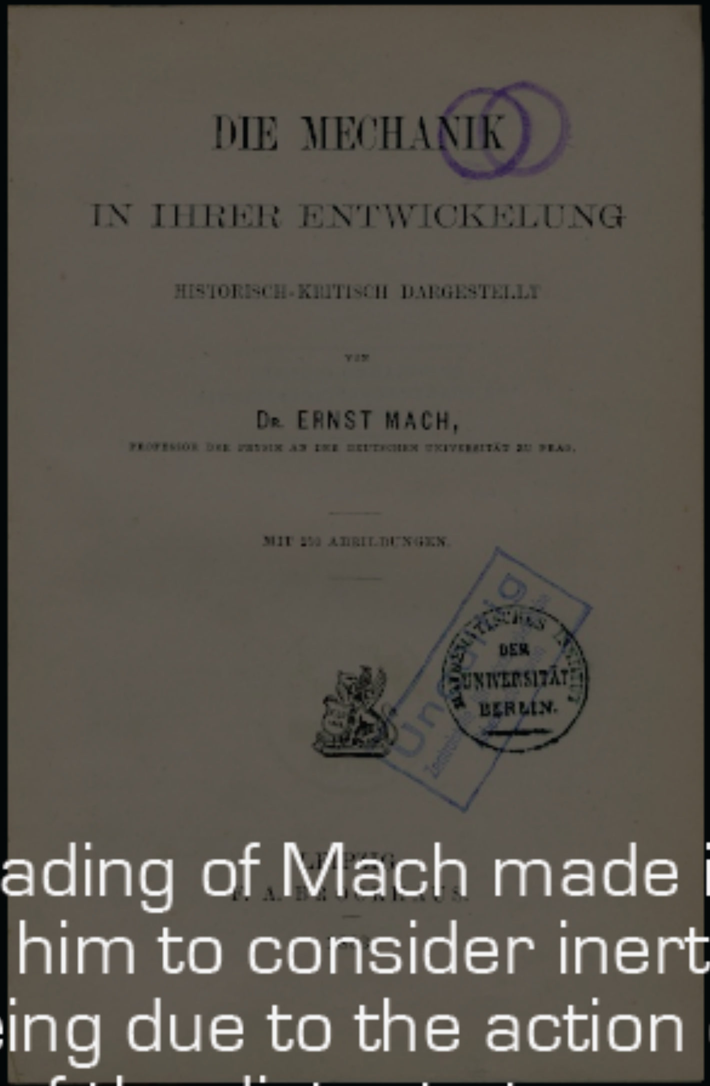
The problem: How to extend special relativity so that it can be applied to all reference systems and not only to inertial ones?

$\boxed{\text{What is it that makes a frame of reference non-inertial?}}$

Einstein, A., 1908: "*Relativitätsprinzip und die aus demselben gezogenen Folgerungen* (On the Relativity Principle and the Conclusions Drawn from It)", Jahrbuch der Radioaktivität (Yearbook of Radioactivity) 4: 411–462

Einstein and Mach




DIE MECHANIK

IN IHRER ENTWICKELUNG

HISTORISCH-KRITISCH DARGESTELLT

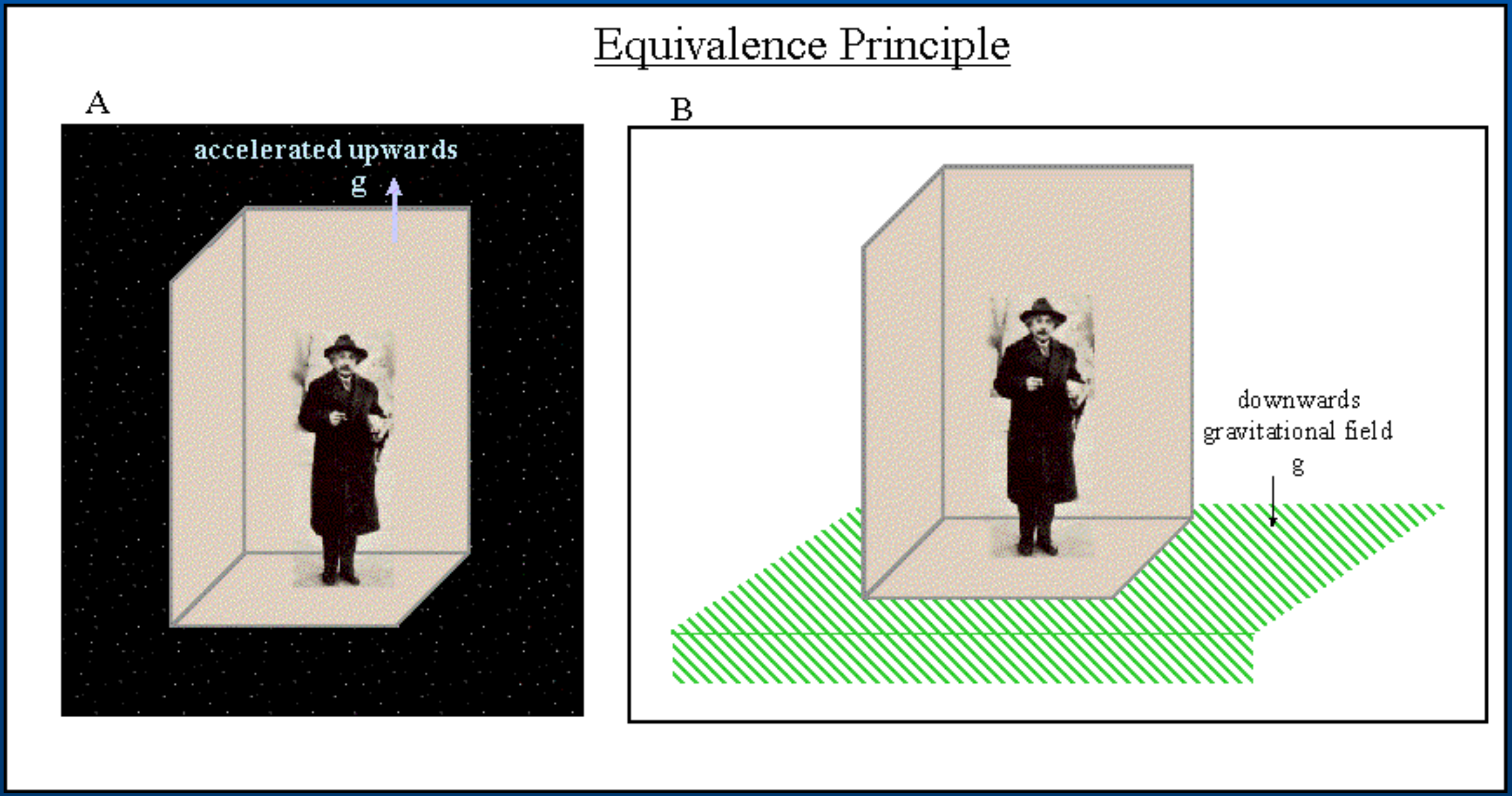
VON

Dr. ERNST MACH,

Einstein's reading of Mach made it plausible for him to consider inertial forces as being due to the action of the masses of the distant stars.

The Equivalence Principle

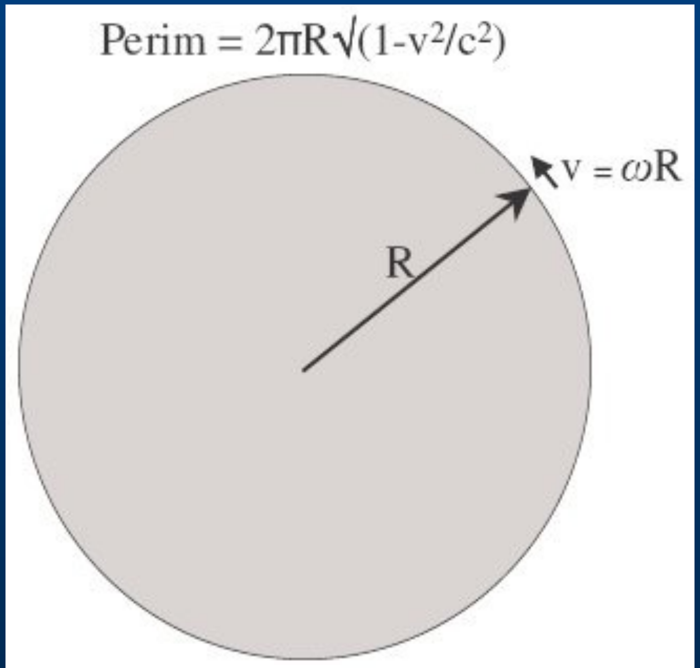
A uniformly accelerated system is indistinguishable from a system under a uniform gravitational field.



So, gravitation and inertia seem to be two aspects of the same phenomenon: a gravito-inertial field.

Is it then possible to describe gravitation by means of a field theory? How to represent that field?

In 1911 Einstein moved to Prague and there, at the German University, he thought hard about the problem. He discovers, using the Equivalence Principle, that the geometry of space cannot be Euclidean in the presence of a gravity.

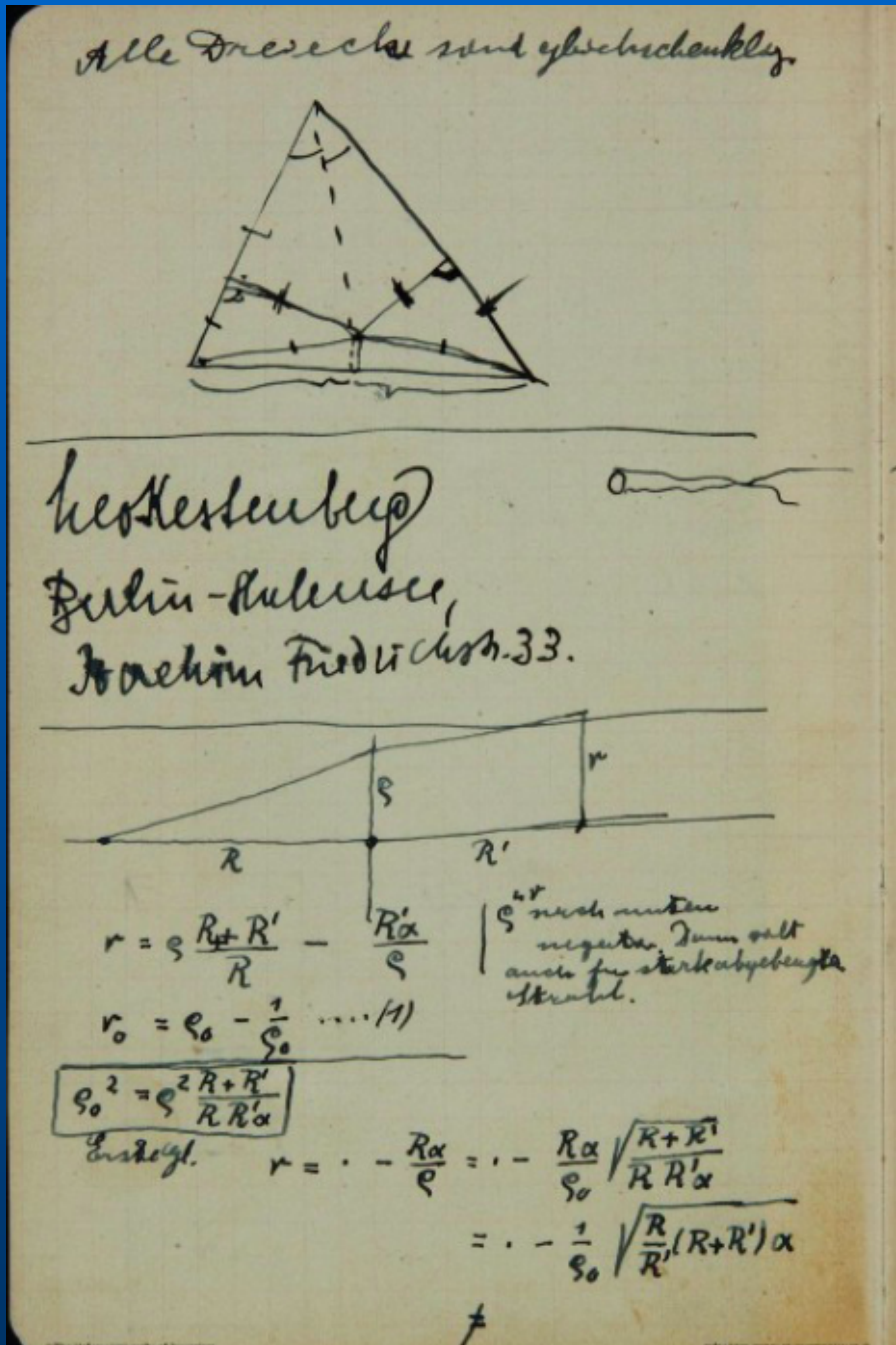


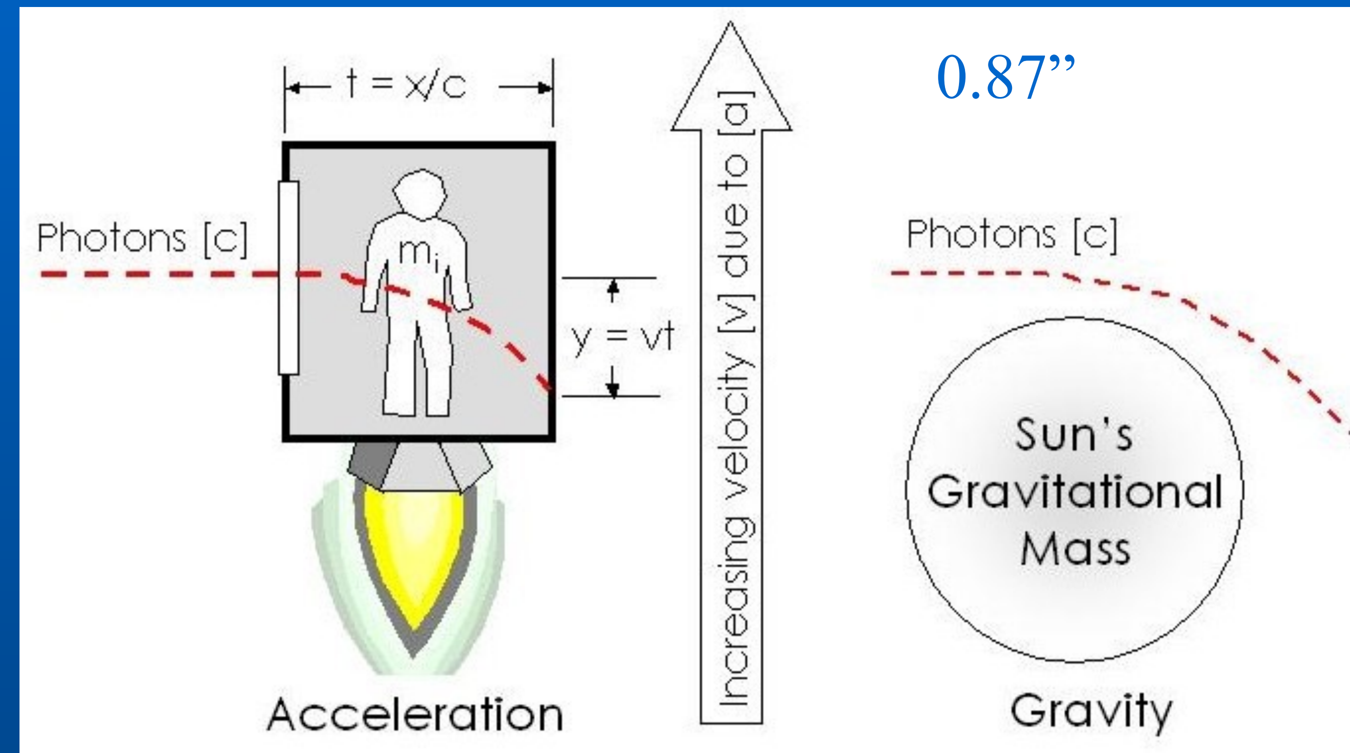
$$\frac{\text{Perimeter}}{\text{Diameter}} = \frac{2\pi R}{\gamma 2R} = \frac{\pi}{\gamma}.$$

$$v = \omega R \rightarrow \gamma = \gamma(R)$$

$$dt = \gamma(R) d\tau$$

Gravitation implies a distortion space and time from Euclidean geometry!



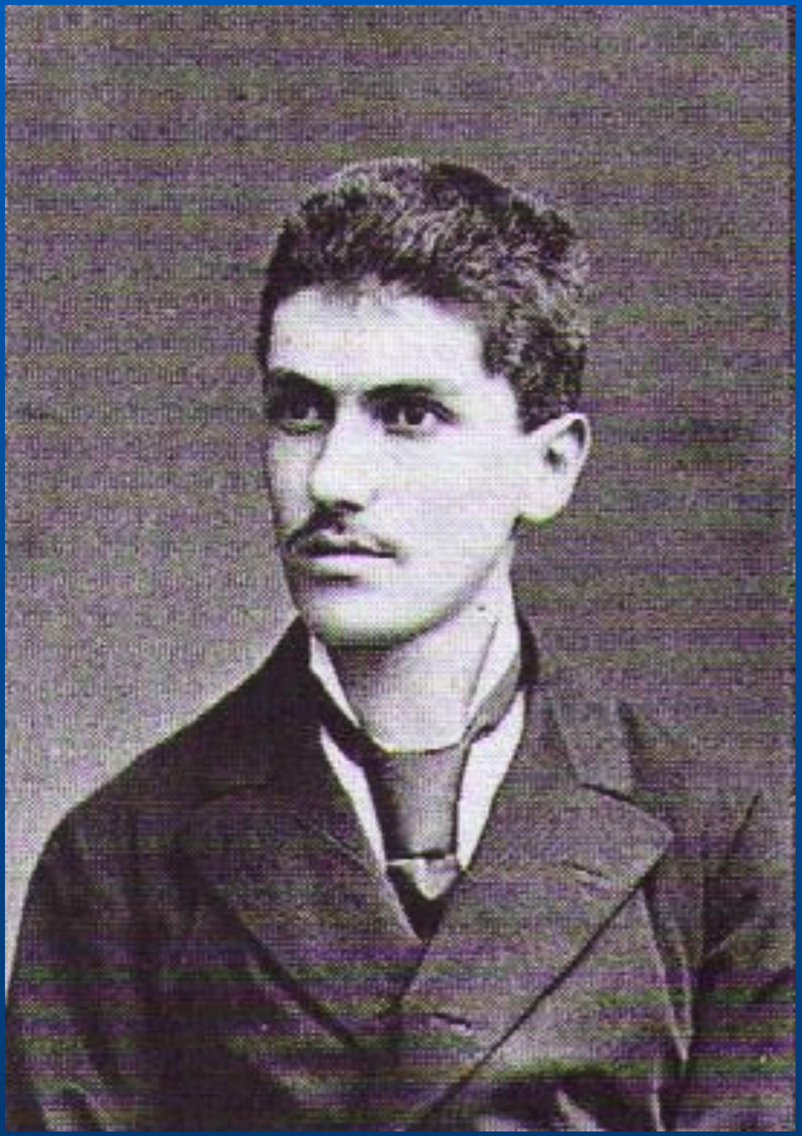


Gravitational lensing!

Albert Einstein: Prague Notebook, 1912

Gravitation affect slight paths!

How could Einstein transform his physical ideas and intuitions based on the Equivalence Principle into a gravitational field theory?

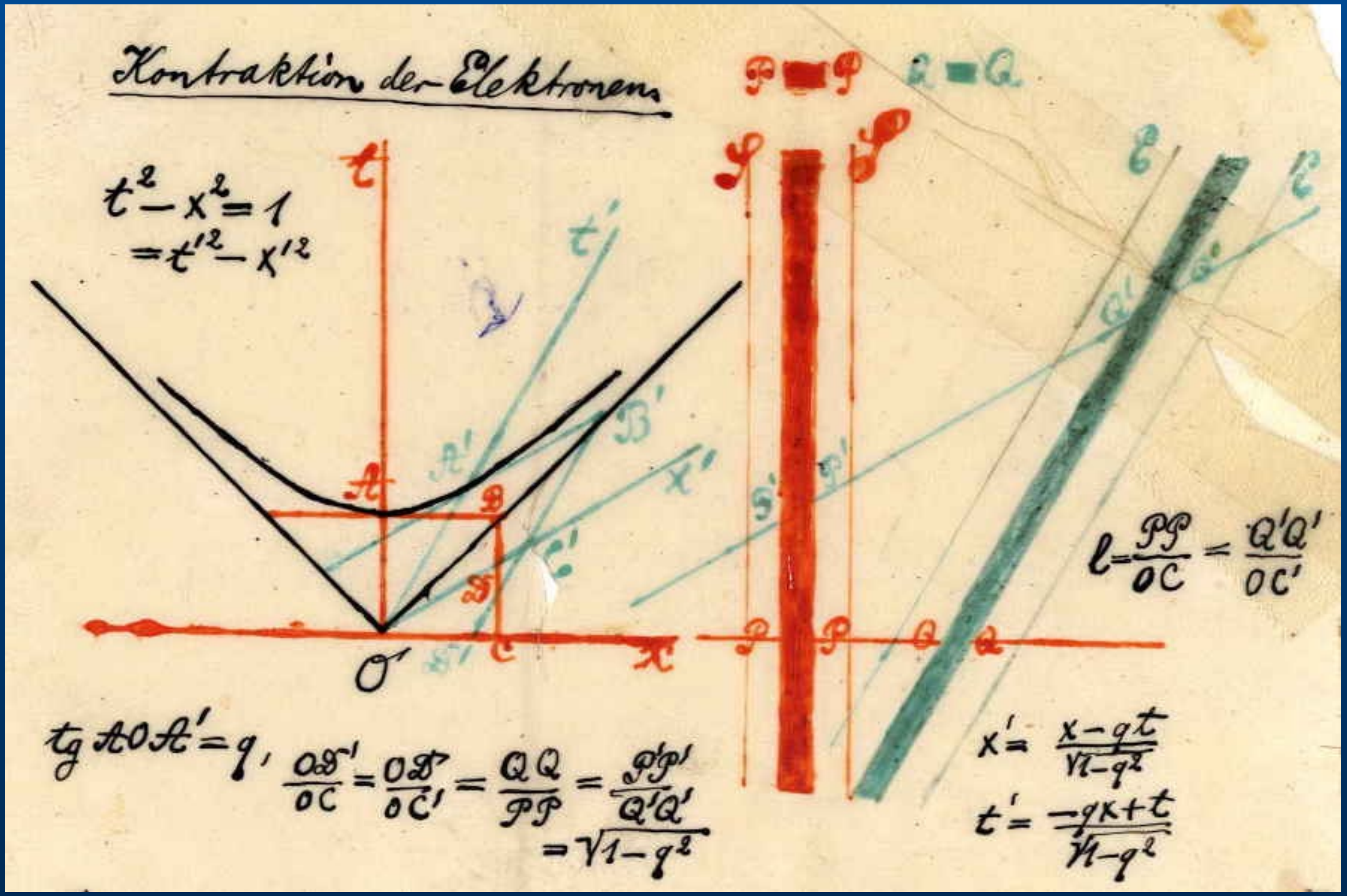


Georg Alexander Pick
(1859-1942)

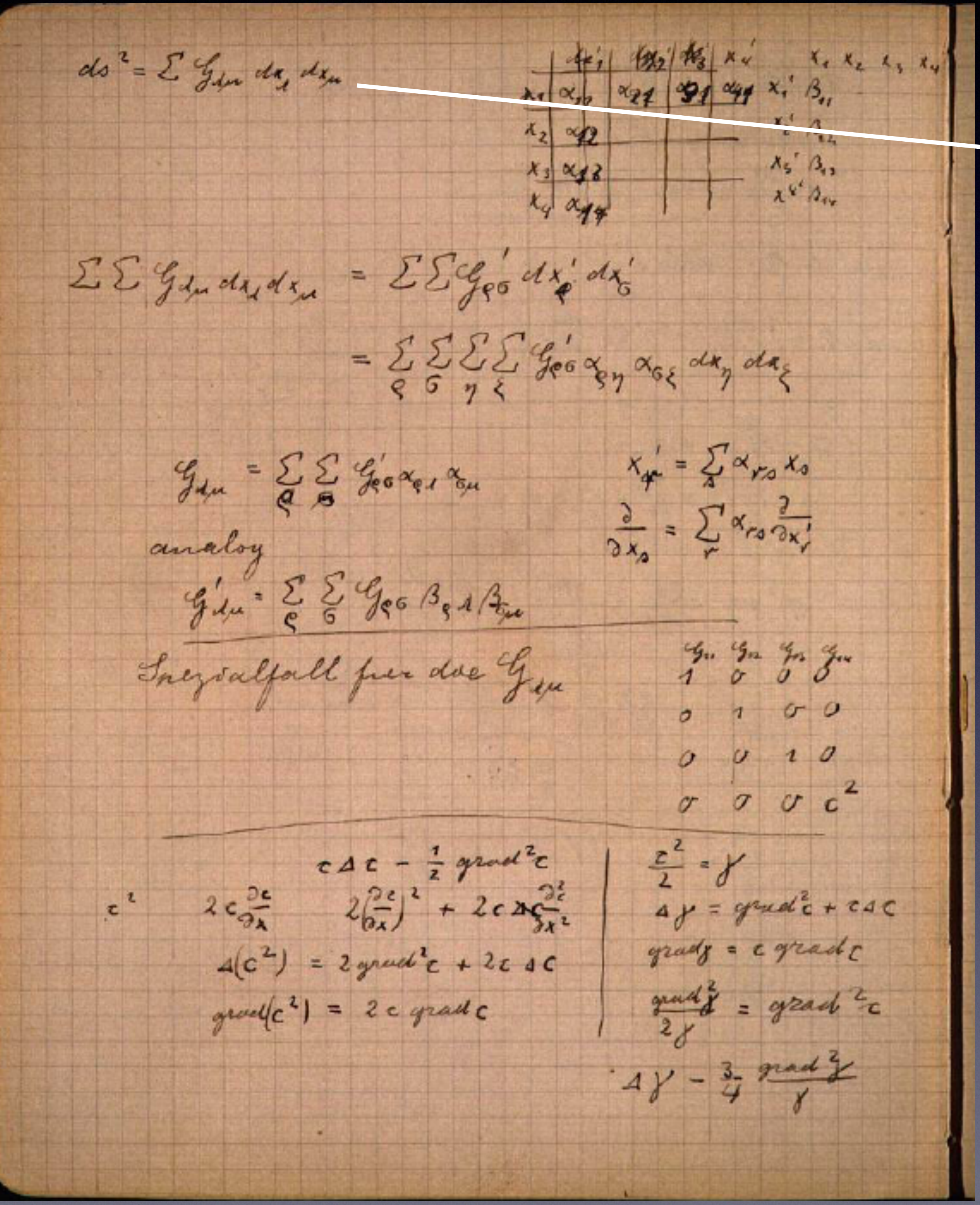
Georg Pick, in Praga, suggests Einstein appealing to tensor analysis.

“The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.”

H. Minkowski, Köln, September 21st, 1908



In 1912 Einstein returns to Zürich as Ordinary Professor and with Marcel Grossmann help studies the work of Riemann, Ricci, Levi-Civitta, en others.



$$ds^2 = g_{ab}\, dx^a\, dx^b$$

$$F(g_{ab}) = k\, T_{ab}$$

Albert Einstein,
Zurich Notebook (1912/13),
p. 39L.

Spacetime

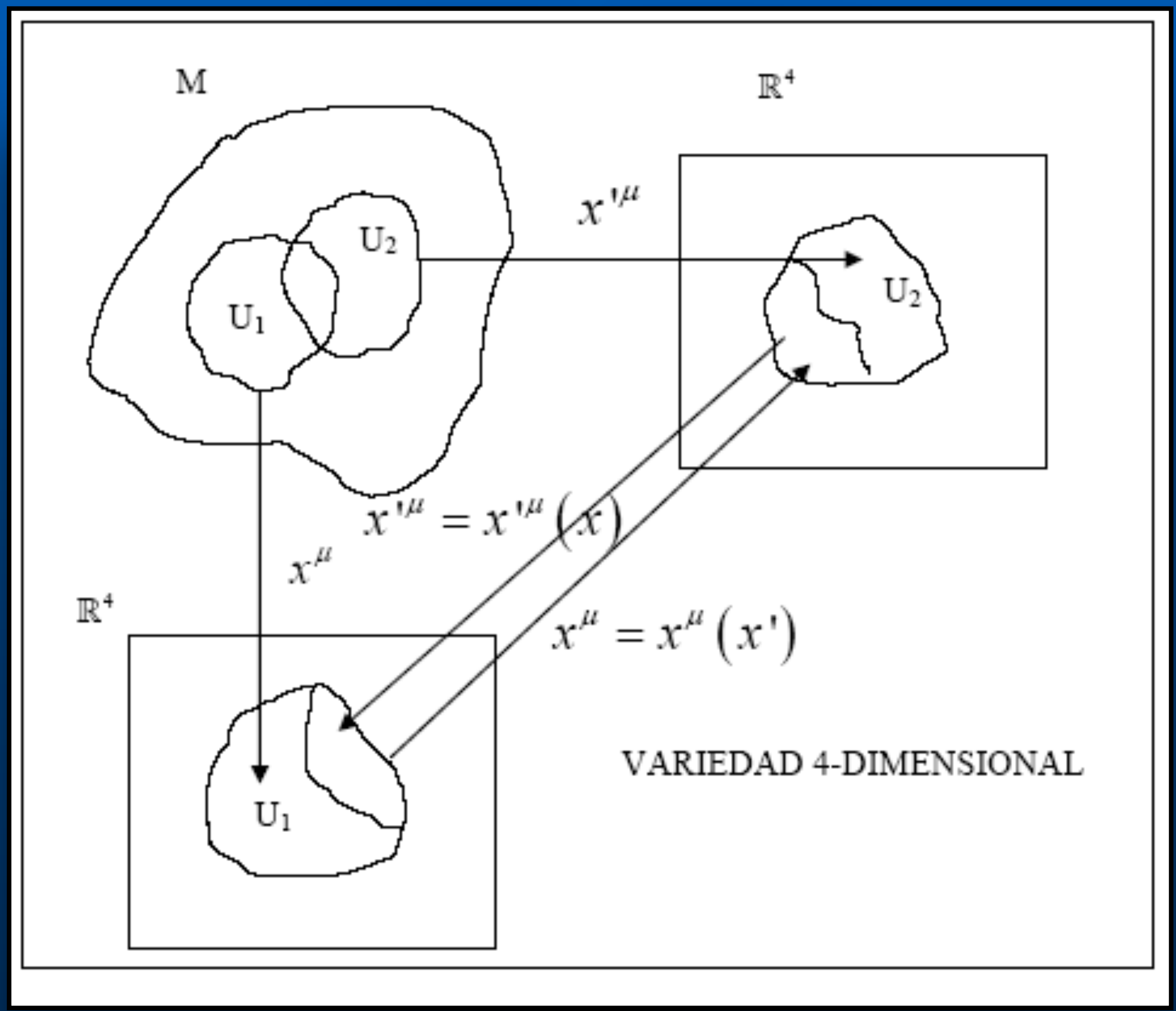
What is spacetime?

Spacetime is a physical system that "includes" both space and time. It is the only system that interacts with all other systems.

How can we represent spacetime?

Spacetime can be represented by a differentiable, 4-dimensional, real manifold.

Spacetime



4-dimensional manifold

$$p \longleftrightarrow \{x^{\mu}\}$$
$$p \longleftrightarrow \{x'^{\mu}\}$$
$$\exists \ \ x'^{\mu} = x'^{\mu}(\{x^{\mu}\})$$

Manifold

A set M is a differentiable manifold if:

1. M is a topological space.
2. M is equipped with a family of pairs $\{(M_\alpha, \varphi_\alpha)\}$.
3. The M_α's are a family of open sets that cover M: $M = \bigcup_\alpha M_\alpha$. The φ_α's are homeomorphisms from M_α to open subsets O_α of $\Re^n$: $\varphi_\alpha : M_\alpha \to O_\alpha$.

4. Given M_α and M_β such that $M_\alpha \cap M_\beta \neq \emptyset$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ from the subset $\varphi_\alpha(M_\alpha \cap M_\beta)$ of $\Re^n$ to the subset $\varphi_\beta(M_\alpha \cap M_\beta)$ of $\Re^n$ is infinitely differentiable (C^∞).

A manifold M is said to be Hausdorff if for any two distinct elements $x \in M$ and $y \in M$, there exist $O_x \subset M$ and $O_y \subset M$ such that $O_x \cap O_y = \emptyset$.

Topological space

Let X be any set and $T = \{X_\alpha\}$ a collection, finite or infinite, of subsets of X. Then (X, T) for a *topological space* iff:

1. $X \in T$.
2. $\emptyset \in T$.
3. Any finite or infinite sub-collection $\{X_1, X_2, \ldots, X_n\}$ of the X_α is such that $\bigcup_1^n X_i \in T$.
4. Any *finite* sub-collection $\{X_1, X_2, \ldots, X_n\}$ of the X_α is such that $\bigcap_1^n X_i \in T$.

The set X is called a topological space and the X_α are called *open sets*. The assignation of T to X is said to "give" a topology to X.

A differentiable manifold is a type of manifold that is locally similar enough to a linear space as to allow to do calculus.

A homeomorphism or topological isomorphism or bi continuous function is a continuous function between topological spaces that has a continuous inverse function.

Objects on the manifold

$$L = \{x^\mu \longrightarrow x'^\mu = L^\mu_\nu x^\nu\}\,,$$

$$L^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\nu}.$$

Objects are defined by the transformation properties

$$\phi(x^\mu) = \phi'(x'^\mu).$$

contravariant

$$A'^\mu = \sum_{\nu=1}^{4} A^\nu \frac{\partial x'^\mu}{\partial x^\nu}.$$

$$A'^\mu = A^\nu \frac{\partial x'^\mu}{\partial x^\nu}.$$

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu.$$

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} B_\nu.$$

$$\frac{\partial \phi}{\partial x'^\nu} = \frac{\partial x^\mu}{\partial x'^\nu}\frac{\partial \phi}{\partial x^\mu}.$$

Objects on the manifold

$$\{x^\mu\} \to \{x'^\mu\} \Rightarrow$$

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}$$

$$T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}$$

$$T'^{\mu}_{\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} T^{\alpha}_{\beta}$$

$$T'^{\overbrace{\cdots\mu\cdots}^{n}}_{\underbrace{\cdots\nu\cdots}_{m}} = \cdots \frac{\partial x'^\mu}{\partial x^\rho} \cdots \frac{\partial x^\sigma}{\partial x'^\nu} \cdots T^{\cdots\rho\cdots}_{\cdots\sigma\cdots}$$

The tensor is *n* times contravariant, *m* times covariant

Tensor field

$$p \longrightarrow T^{\cdots\mu\cdots}_{\cdots\nu\cdots}(p),$$

Spacetime: metric

We need to know how to measure distances over a manifold. These distances are the intrinsic separation between events of spacetime. We do this introducing a metric tensor. Spacetime, then, is represented by an order pair (M, g), where g is the metric tensor.

Euclidean metric

$$\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Interval

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Minkowski metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

The Minkowski metric tensor $\eta_{\mu\nu}$ has rank 2 and trace -2. We call the coordinates with the same sign *spatial coordinates* (adopting the convention $x^1 = x$, $x^2 = y$, and $x^3 = z$) and the coordinate $x^0 = ct$ is called *temporal coordinate*. The constant c is introduced to make the units uniform.

Minkowski Spacetime

Minkowski metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Interval

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2. \end{aligned}$$

Proper time

$$d\tau^2 = \frac{1}{c^2} ds^2.$$

$$\begin{aligned} \mathrm{d}\tau^2 &= \frac{1}{c^2}(c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2) \\ &= \mathrm{d}t^2 \left\{1 - \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right]\right\} \\ &= \mathrm{d}t^2\left(1 - \frac{v^2}{c^2}\right), \end{aligned}$$

$$d\tau = \frac{dt}{\gamma},$$

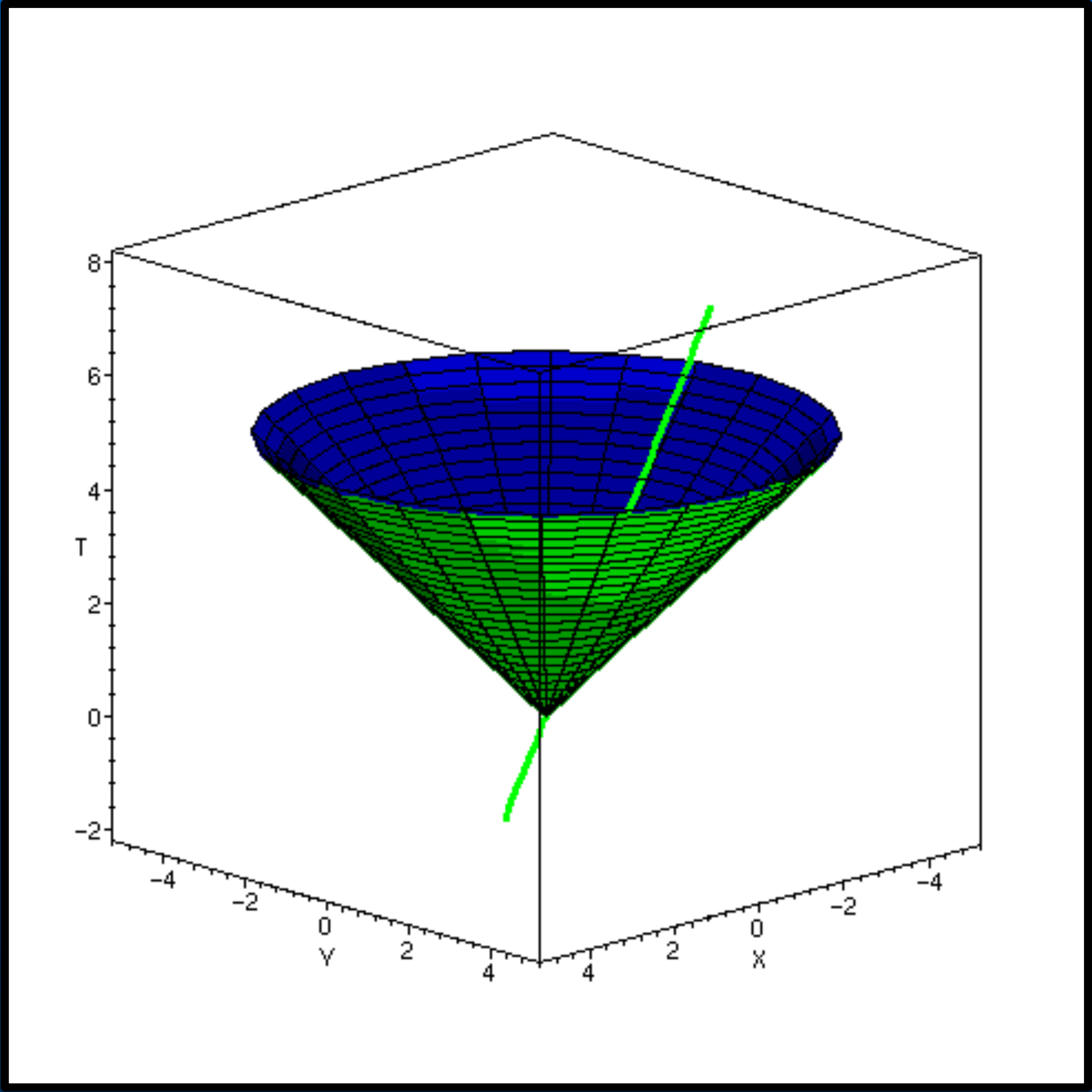
$$\gamma = \frac{1}{\sqrt{1 - (\frac{v}{c})^2}}$$

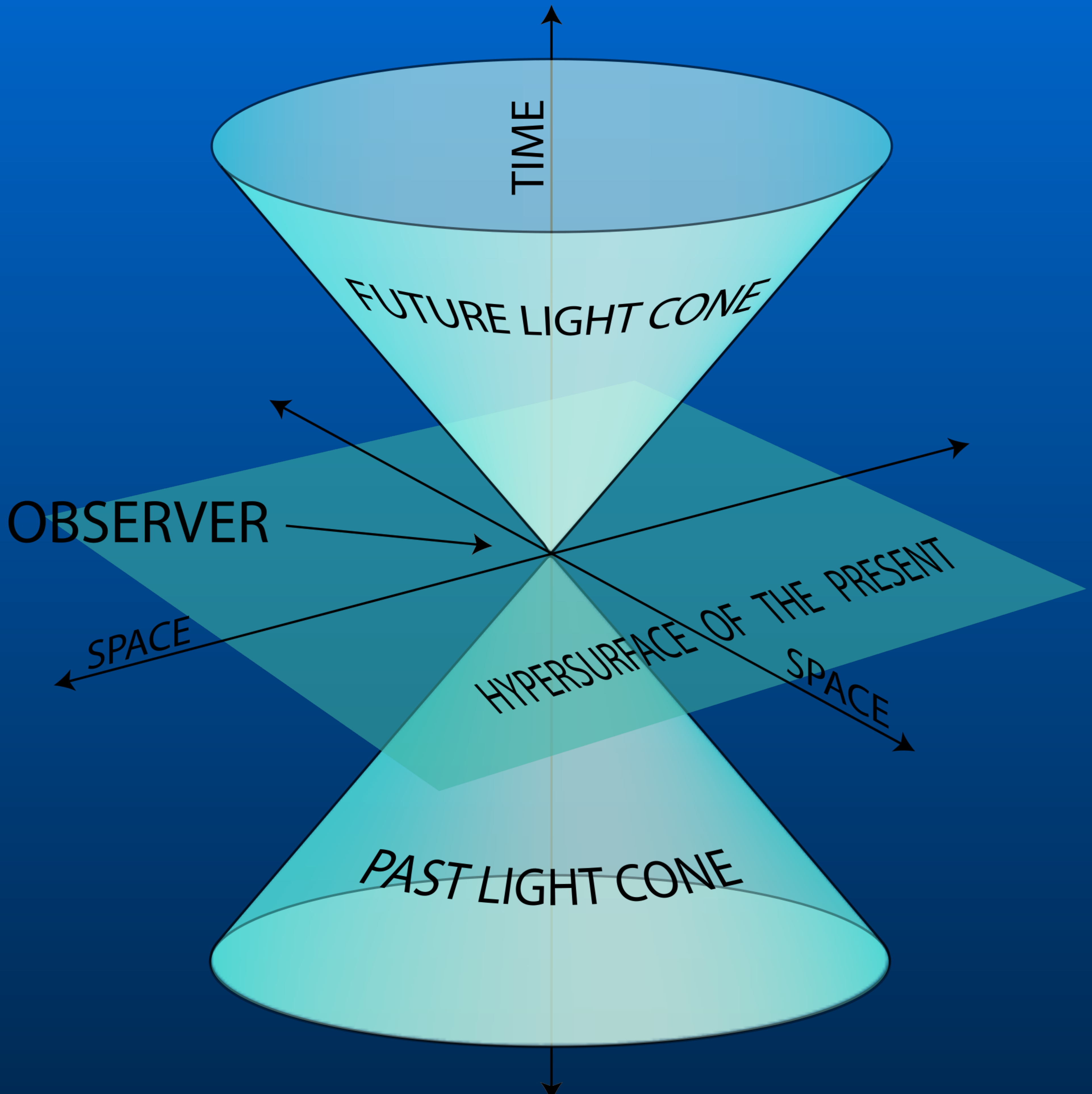
Minkowski Spacetime

There is a partial ordering of events. Simultaneity is *not* absolute in spacetime

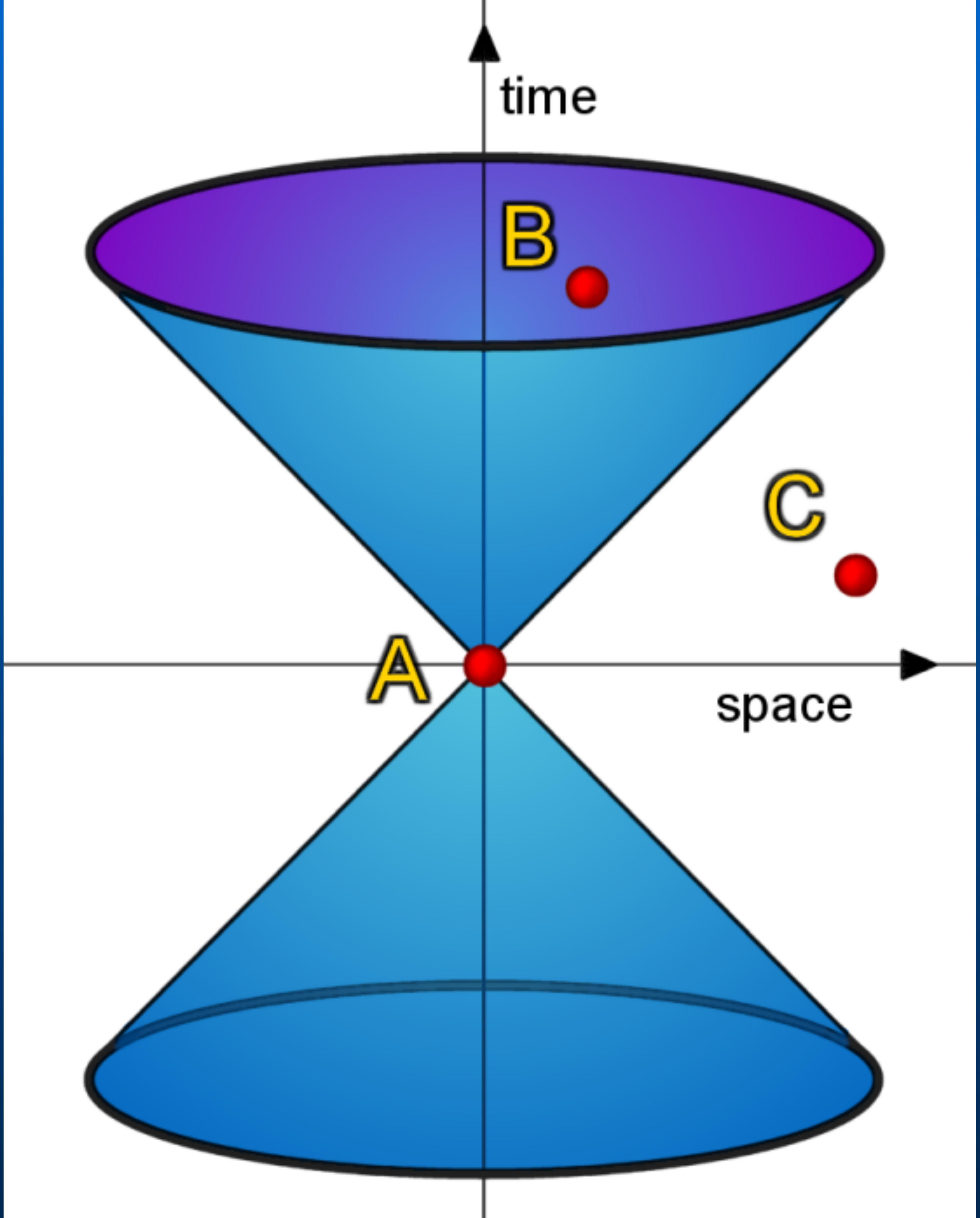
for $ds^2 > 0$,	the interval is timelike;
for $ds^2 = 0$,	the interval is null or lightlike;
for $ds^2 < 0$,	the interval is spacelike.

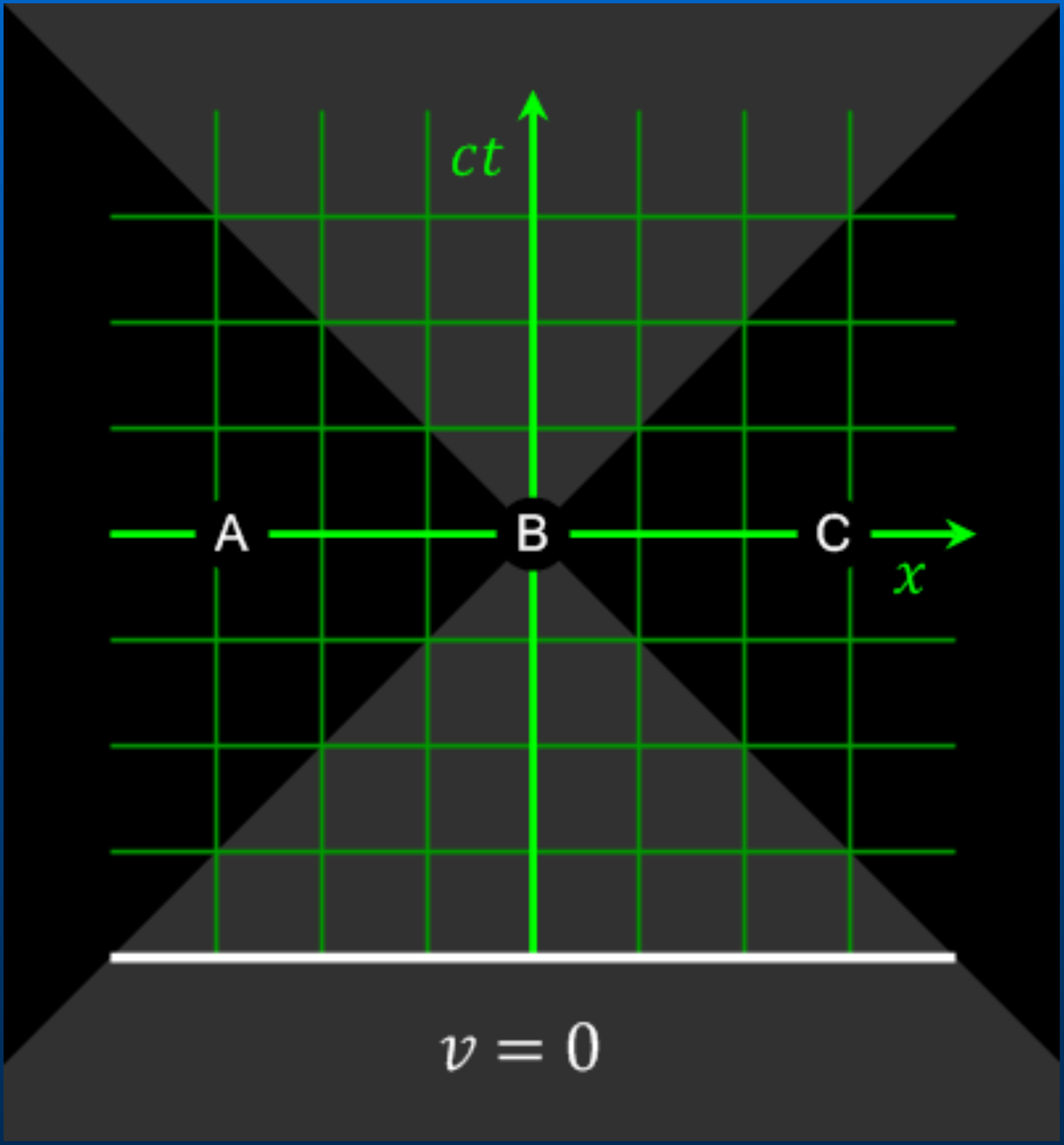
Light cones





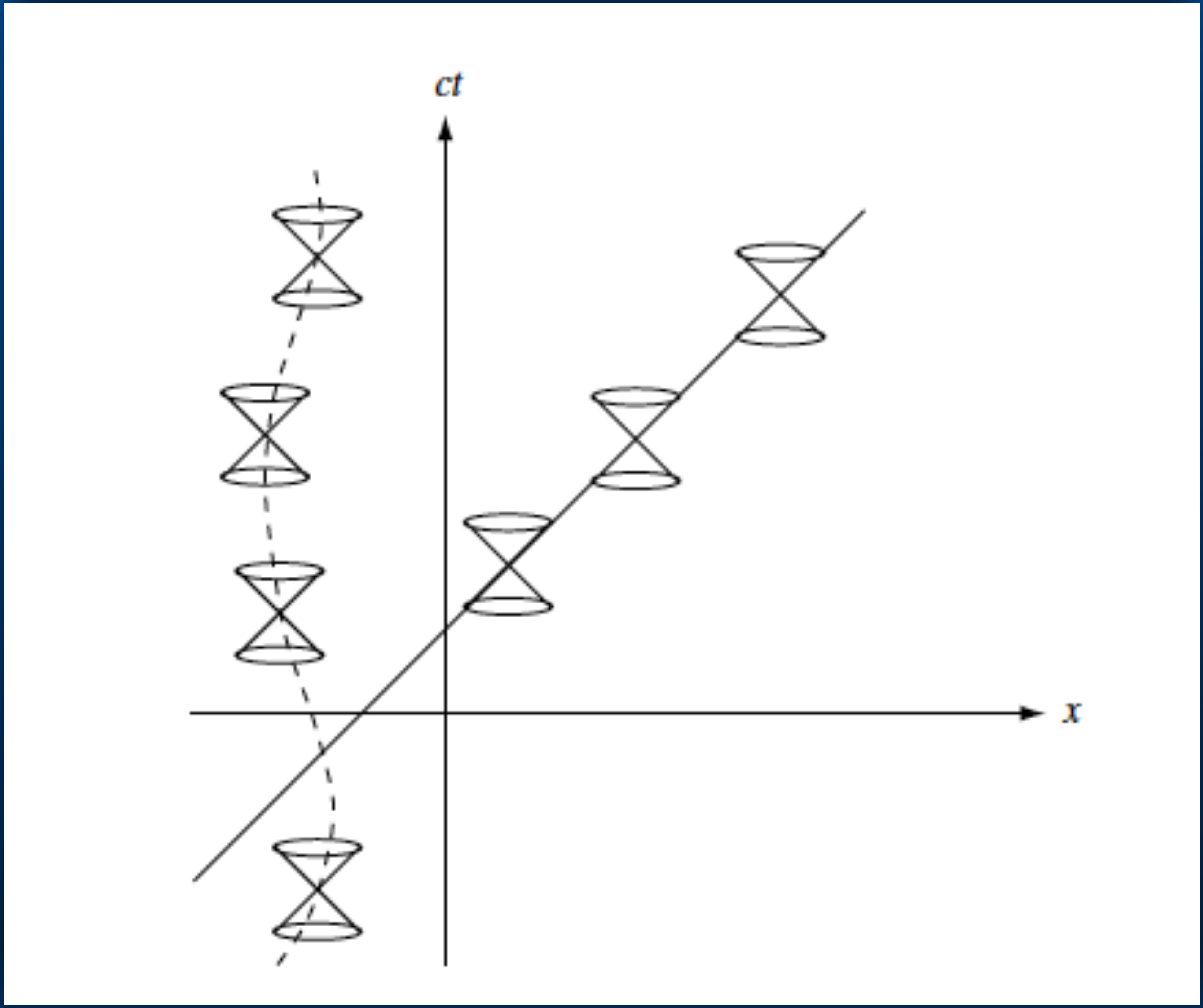
TIME
FUTURE LIGHT CONE
OBSERVER
SPACE
HYPERSURFACE OF THE PRESENT
SPACE
PAST LIGHT CONE


time
B
C
A
space

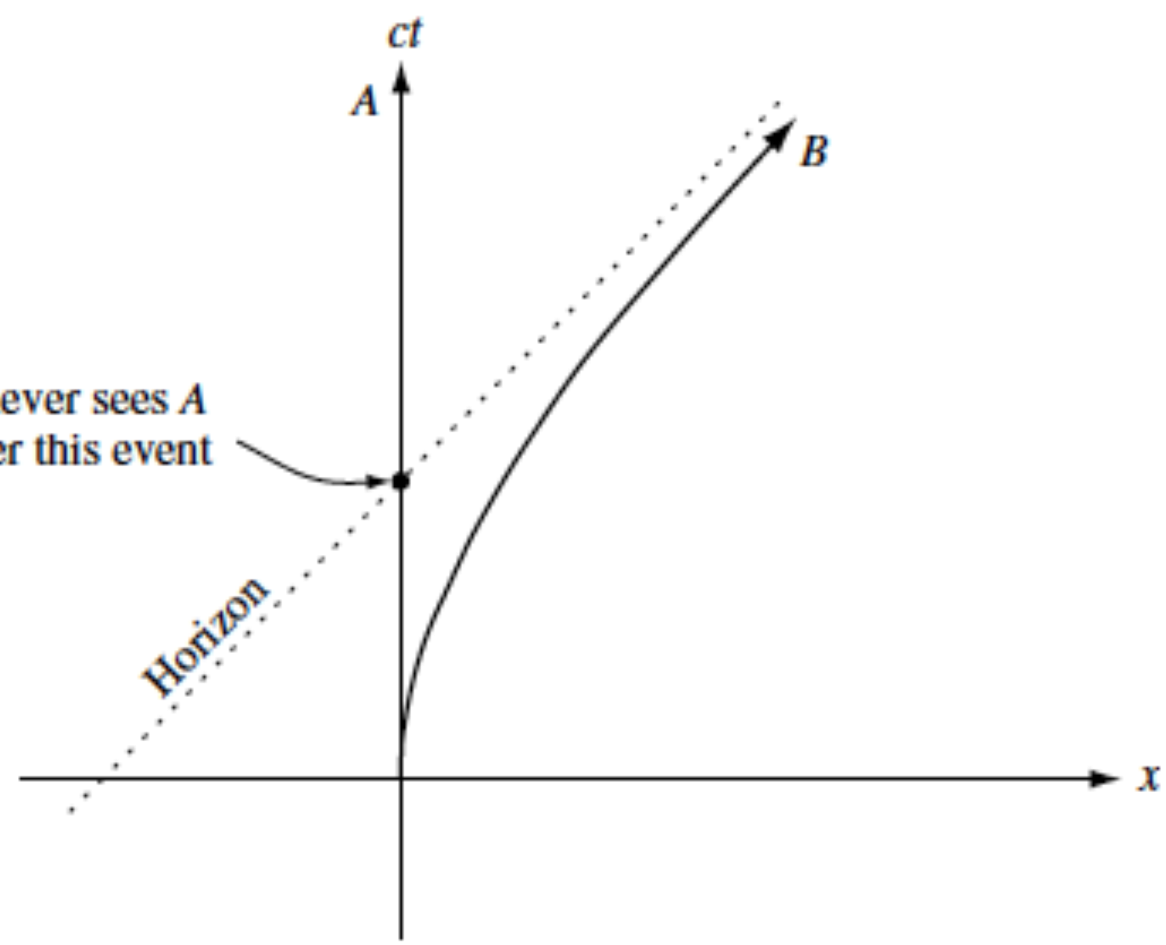


ct
A
B
C
x
$v = 0$

Light cones

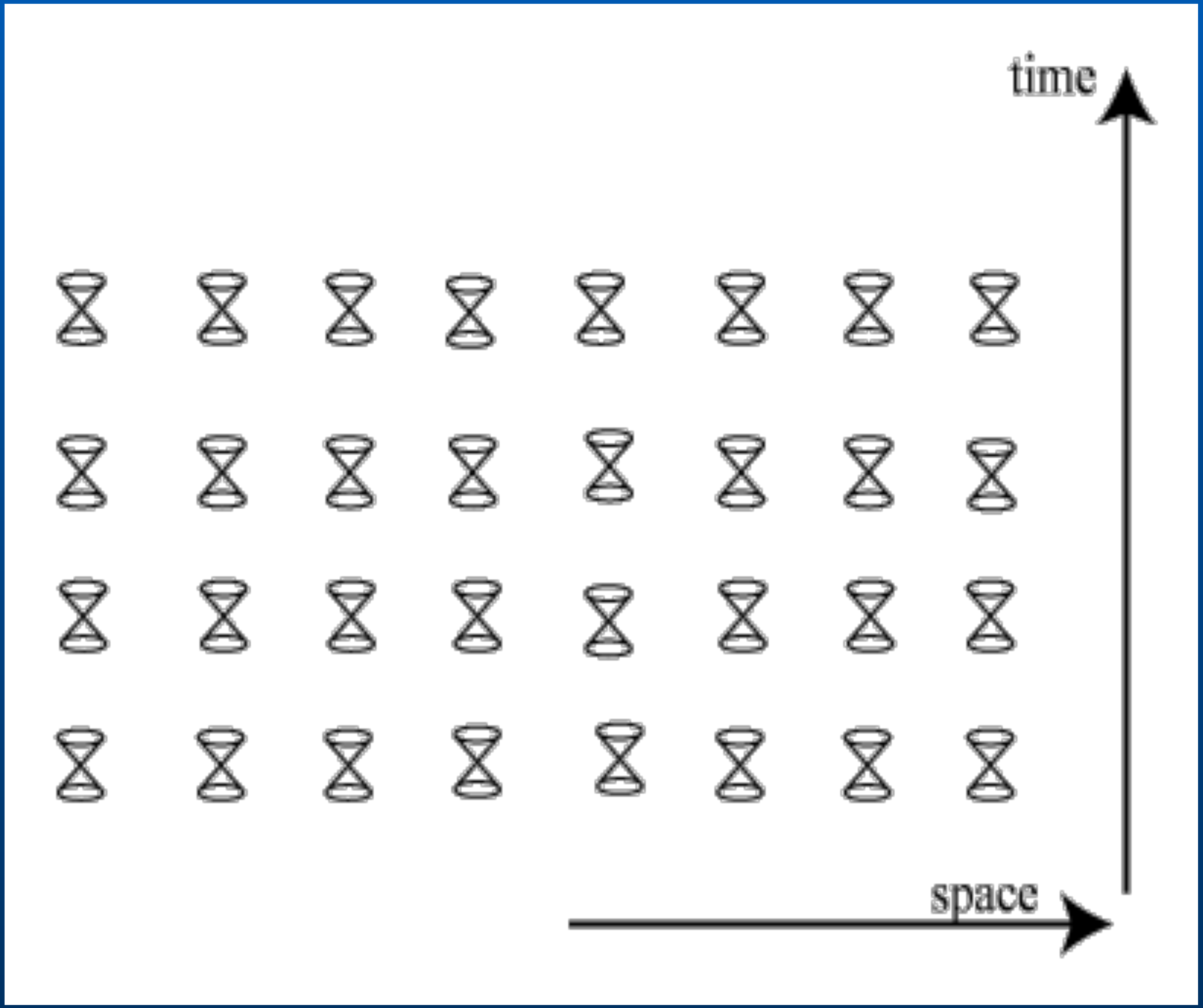


Particle horizon



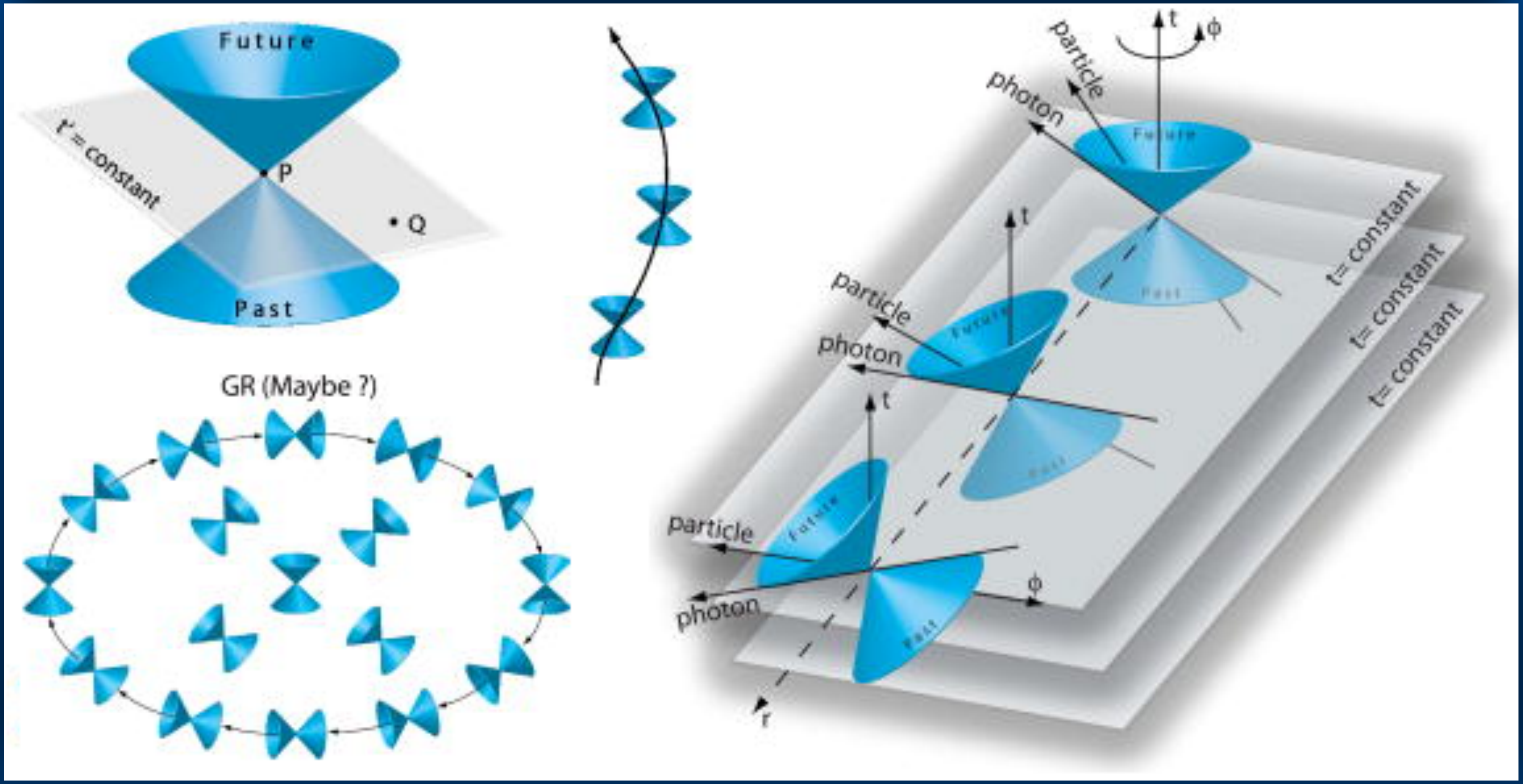
The worldline of a uniformly accelerated particle B starting from rest from the origin of S. If an observer A remains at $x = 0$, then the worldline of A is simply the t-axis. No message sent by A after $t = c/f$ will ever reach B.

Light cones

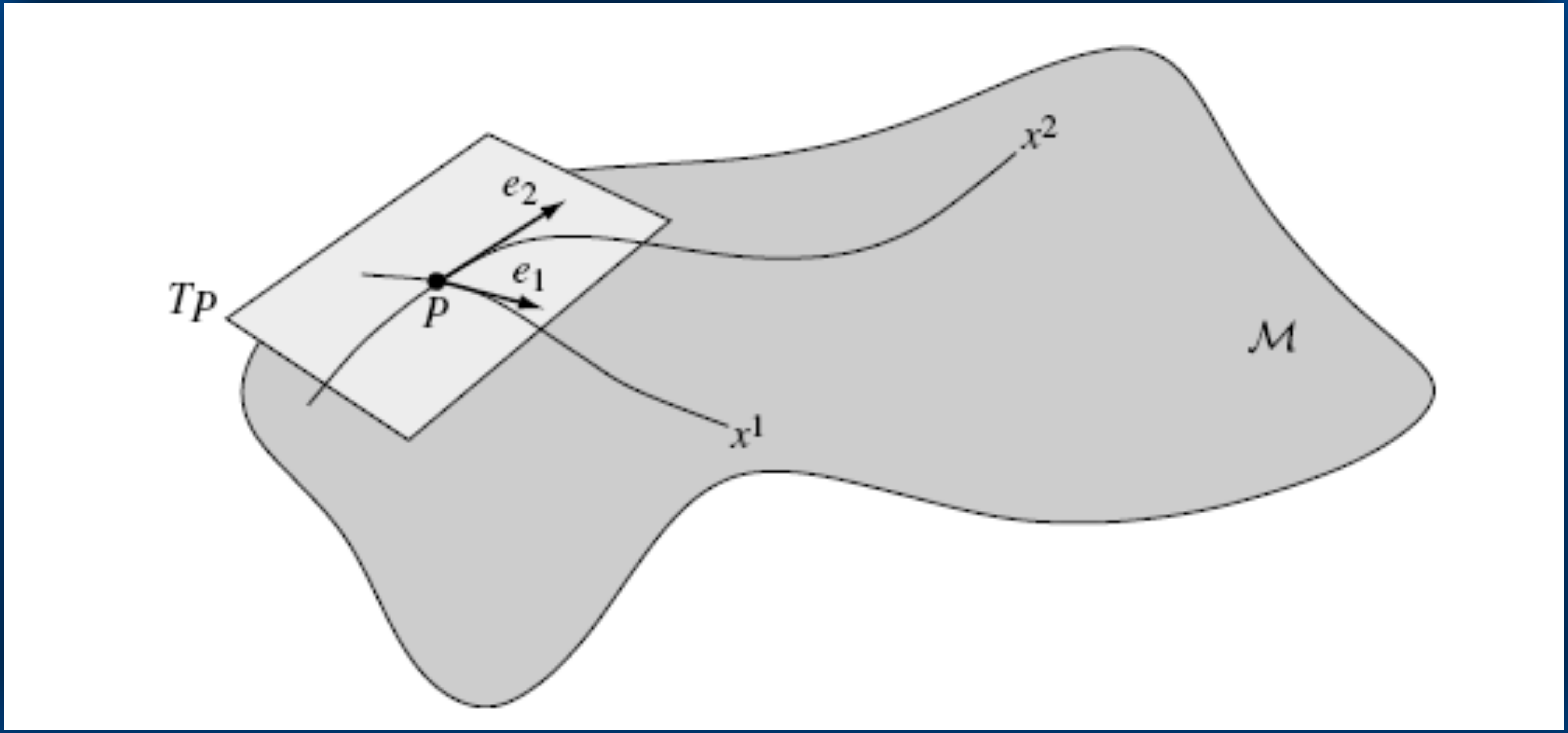


More general space-times

$$g_{\mu\nu} = g_{\mu\nu}(x)$$



Tetrads



The coordinate basis vectors $\boldsymbol{e}_a$ at a point P in a manifold are the tangent vectors to the coordinate curves in the manifold and form a basis for the tangent space at P.

$$\boldsymbol{v}(x) = v^a(x)\ \boldsymbol{e}_a(x).$$

For any set of basis vectors $e_a(x)$, we can define a second set of vectors called the dual basis vectors:

$$d\boldsymbol{s} = \boldsymbol{e}_a(x)\, dx^a.$$

$$ds^2 = d\boldsymbol{s} \cdot d\boldsymbol{s} = (dx^a \boldsymbol{e}_a) \cdot (dx^b \boldsymbol{e}_b) = (\boldsymbol{e}_a \cdot \boldsymbol{e}_b)\, dx^a\, dx^b.$$

$$ds^2 = g_{ab}(x)\, dx^a\, dx^b.$$

$$\boldsymbol{e}_a(x) \cdot \boldsymbol{e}_b(x) = g_{ab}(x).$$

At any given point P the manifold is flat, so

$$g_{\mu\nu}(P) = \eta_{\mu\nu}.$$

A manifold with such a property is called *pseudo-Riemannian*. If $g_{\mu\nu}(P) = \delta_{\mu\nu}$ the manifold is called strictly *Riemannian*.

The basis is called *orthonormal* when $\hat{e}^{\mu} \bullet \hat{e}_{\nu} = \eta^{\mu}_{\nu}$ at any given point P. Notice that since the tetrads are 4-dimensional we can write

$$e_{\mu a}(x) e^{a}_{\nu}(x) = g_{\mu\nu}(x),$$

and

$$e_{\mu a}(P) e^{a}_{\nu}(P) = \eta_{\mu\nu}.$$

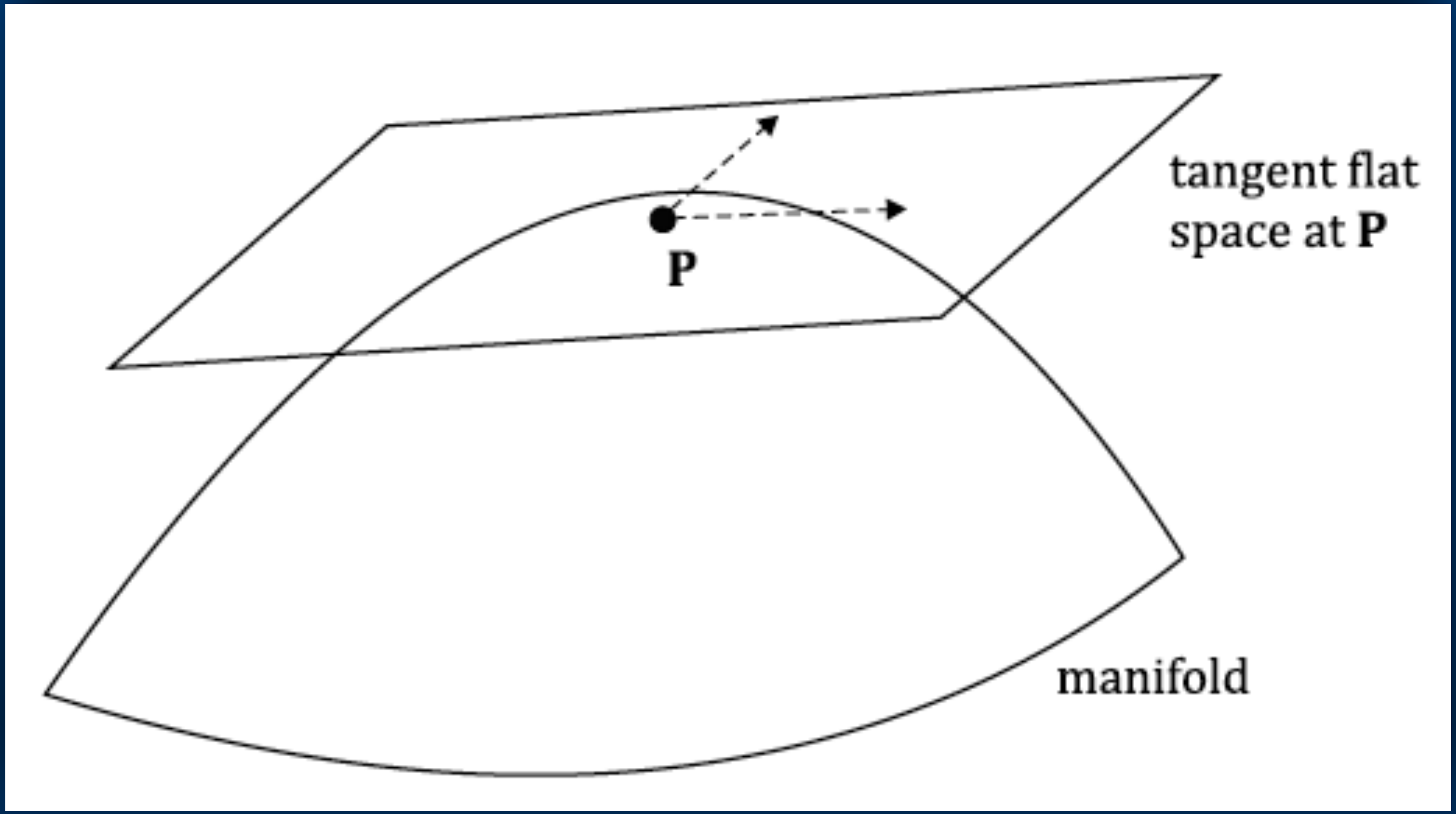
Equivalence principle

In an arbitrary spacetime it is always possible to find a reference system such that, locally, all laws of physics can be expressed in it as those valid for Minkowskian spacetime.

Pseudo-Riemannian spaces

In order to introduce gravitation in a general space-time we define a metric tensor $g_{\mu\nu}$, such that its components can be related to those of a locally Minkowski space-time defined by $ds^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta$ through a general transformation:

$$ds^2 = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu .$$



Pseudo-Riemannian spaces

$$\frac{d^2\xi^\alpha}{ds^2} = 0, \quad \Longrightarrow \quad \begin{aligned} \frac{d}{ds}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{ds}\right) &= 0 \\ \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{ds^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} &= 0. \end{aligned}$$

Now, multiplying at both sides by $\partial x^\lambda / \partial \xi^\alpha$ and using:

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\lambda}{\partial \xi^\alpha} = \delta^\lambda_\mu,$$

we get

$$\frac{d^2 x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0,$$

where $\Gamma^\lambda_{\mu\nu}$ is the affine connection of the manifold:

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}.$$

Geodesic equation

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}=0.$$

or

$$\ddot{x}^a+\Gamma^a{}_{bc}\dot{x}^b\dot{x}^c=0.$$

If there are non-gravitational forces:

$$\ddot{x}^a+\Gamma^a{}_{bc}\dot{x}^b\dot{x}^c=-g^{ab}\partial_b V.$$

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu{}_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}=\frac{q}{m_0}F^\mu{}_\nu\frac{dx^\nu}{d\tau}.$$

$$\Gamma'^{a}_{bc} = \frac{\partial x'^{a}}{\partial x^{d}} \frac{\partial x^{f}}{\partial x'^{b}} \frac{\partial x^{g}}{\partial x'^{c}} \Gamma^{d}{}_{fg} - \frac{\partial x^{d}}{\partial x'^{b}} \frac{\partial x^{f}}{\partial x'^{c}} \frac{\partial^{2} x'^{a}}{\partial x^{d} \partial x^{f}}.$$

It is not a tensor!

The geodetic equation can be obtained independently of the Equivalence Principle. We can assume a more general principle: any spacetime trajectory of a free system is minimal.

$$S = a \int \mathrm{d}\tau,$$

where τ is the proper time and a is a constant with the dimensions of energy. Writing this in general coordinates, we have:

$$S = a \int \sqrt{g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu},$$

$$S = a \int \mathrm{d}\lambda \, \frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \int \mathrm{d}\lambda \sqrt{g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}.$$

$$\delta S = a \int \mathrm{d}\lambda \, \frac{1}{2} \frac{\mathrm{d}\lambda}{\mathrm{d}\tau} \left\{ \delta g_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} + 2 g_{\mu\nu} \frac{\mathrm{d}\delta x^{\mu}}{\mathrm{d}\lambda} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\lambda} \right\} = 0.$$

$$\delta g_{\mu\nu} = (\partial_{\lambda} g_{\mu\nu}) \, \delta x^{\lambda};$$

$$\delta S = \frac{a}{2} \int \mathrm{d}\tau \left\{ (\partial_{\lambda} g_{\mu\nu}) \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} - 2 (\partial_{\rho} g_{\mu\nu}) \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} g_{\mu\lambda} - 2 g_{\mu\nu} \frac{\mathrm{d}^2 x^{\nu}}{\mathrm{d}\tau^2} g_{\mu\lambda} \right\} \delta x^{\lambda} = 0.$$

$$\delta S = a \int \left\{ -\Gamma^{\lambda}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} - \frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau^2} \right\} g_{\lambda\sigma} \delta x^{\sigma} \mathrm{d}\tau = 0.$$

$$\rightarrow \quad \frac{\mathrm{d}^2 x^{\lambda}}{\mathrm{d}\tau^2} + \Gamma^{\lambda}_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{x^{\nu}}{\mathrm{d}\tau} = 0.$$

Derivatives

The usual derivative is not tensor

$$A'^{\mu}_{,\nu} = \frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x'^{\mu}}{\partial x^{\mu}} A^{\mu}\right) = \frac{\partial x'^{\mu}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x'^{\nu}} A^{\mu}_{,\nu} + \frac{\partial^2 x'^{\mu}}{\partial x^{\mu}\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial x'^{\nu}} A^{\mu}.$$

$$A_{\mu;\nu} = \frac{\partial A_{\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu} A_{\lambda}.$$

$$\nabla_{\nu} A_{\mu} = \frac{\partial A_{\mu}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu} A_{\lambda}.$$

A covariant derivative of a vector field is a rank 2 tensor of type (1, 1). The covariant divergence of a vector field yields a scalar field:

$$\nabla_\mu A^\mu = \partial_\mu A^\mu(x) - \Gamma^\mu_{\alpha\mu} A^\alpha(x) = \phi(x).$$

A tangent vector satisfies $V^\nu V_{\nu;\mu} = 0.$

Covariant derivative

The covariant derivative possesses the following properties:

1. Linearity: For constants a and b one has $\nabla_\mu(aA^{\cdots}_{\cdots} + bB^{\cdots}_{\cdots}) = a\nabla_\mu A^{\cdots}_{\cdots} + b\nabla_\mu B^{\cdots}_{\cdots}$.
2. *Leibnitz rule*: $\nabla_\mu(A^{\cdots}_{\cdots}B^{\cdots}_{\cdots}) = \nabla_\mu(A^{\cdots}_{\cdots})B^{\cdots}_{\cdots} + A^{\cdots}_{\cdots}\nabla_\mu(B^{\cdots}_{\cdots})$.
3. Commutativity with contraction: $\nabla_\mu(A^{\ldots\beta\ldots}_{\ldots\beta\ldots}) = (\nabla_\mu A)^{\ldots\beta\ldots}_{\ldots\beta\ldots}$.
4. For a scalar field: $\nabla_\mu\varphi = \varphi_{,\mu}$.
5. Torsion free: $\nabla_\mu\nabla_\nu\varphi = \nabla_\nu\nabla_\mu\varphi$.
6. $\nabla_\mu\, g_{\alpha\beta} = 0$.

$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu} A_\lambda .$$

$$g_{\alpha\beta;\gamma} = 0.$$

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$$

$$\Rightarrow \quad \boxed{\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad} (\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}).}$$

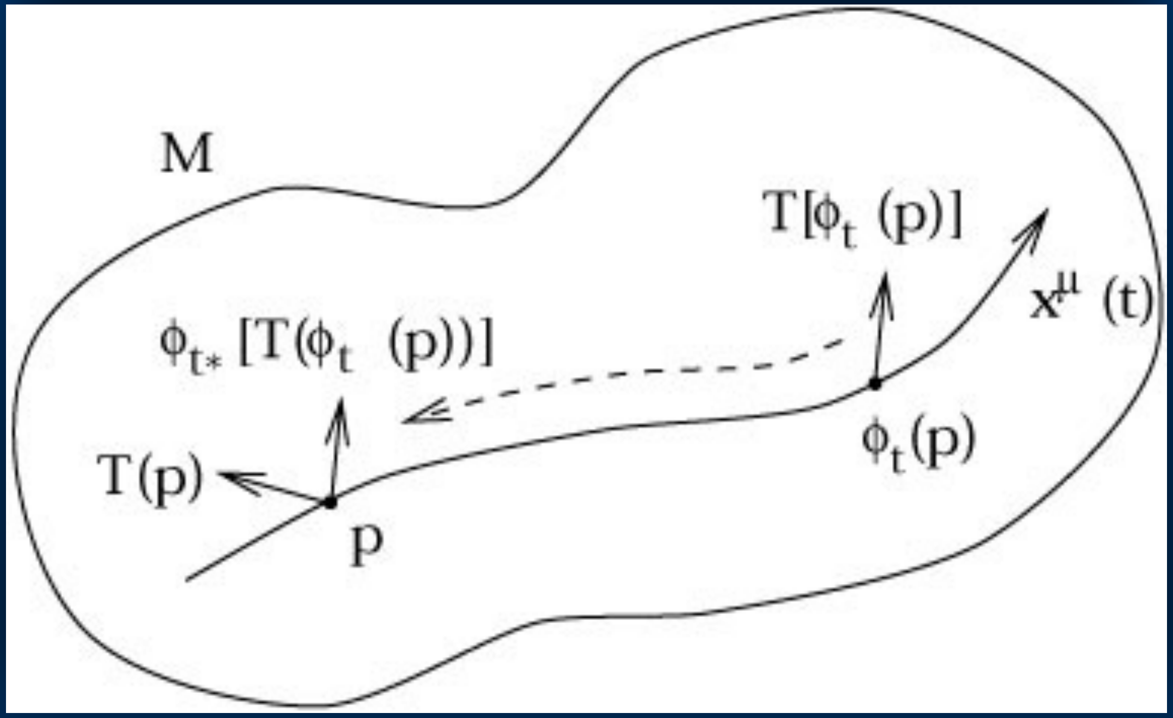
Lie derivatives

If there is a curve γ on the manifold, such that its tangent vector is $u^\alpha = dx^\alpha/d\lambda$ and a vector field A^α is defined in a neighborhood of γ, we can introduce a derivative of A^α along γ as

$$\ell_u A^\alpha = A^\alpha_{,\beta}\, u^\beta - u^\alpha_{,\beta}\, A^\beta = A^\alpha_{;\beta}\, u^\beta - u^\alpha_{;\beta}\, A^\beta .$$

This derivative is a tensor, and it is usually called *Lie derivative*. It can be defined for tensors of any type. A Killing vector field is such that

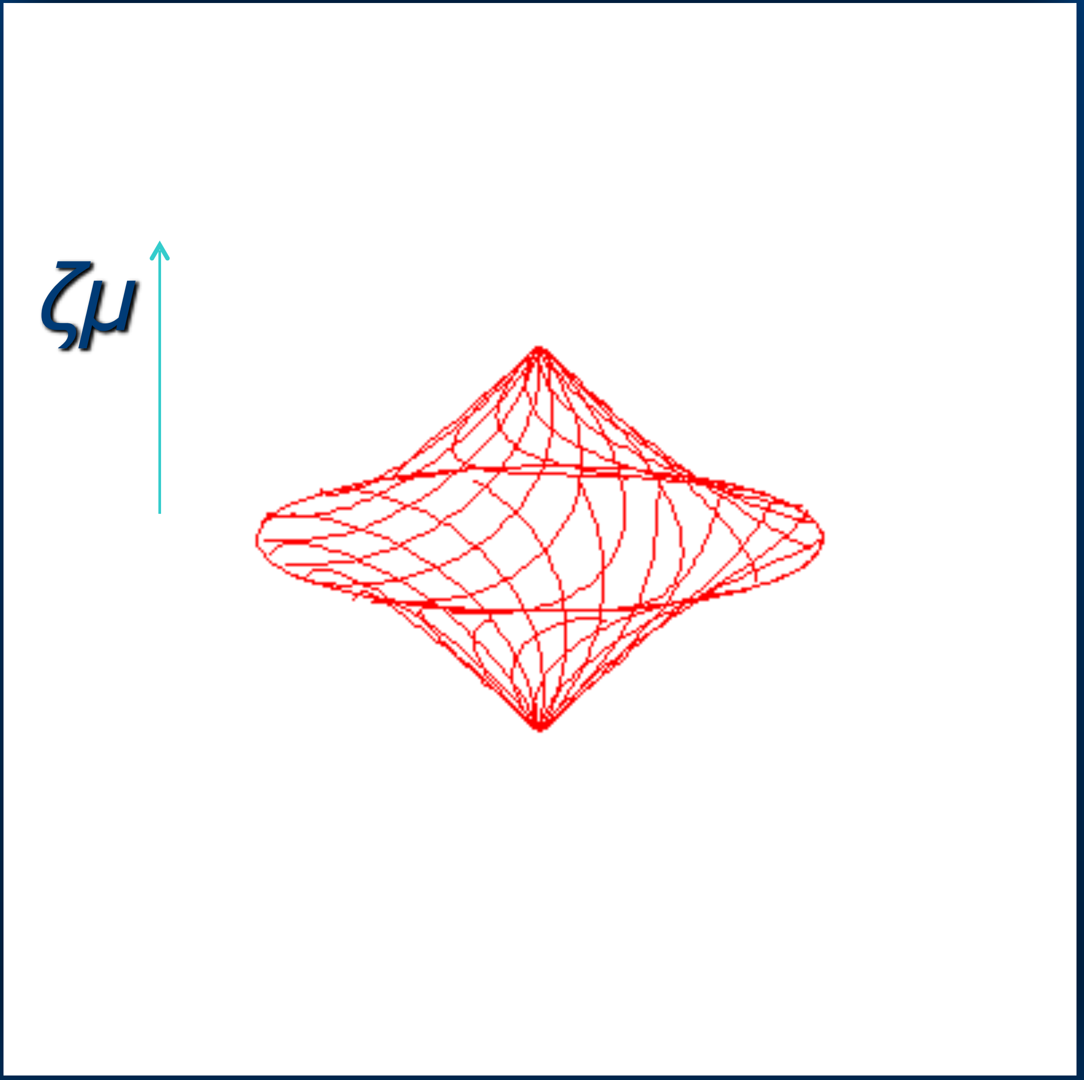
$$\ell_\zeta g_{\mu\nu} = 0.$$



$$0 = \pounds_{\xi} g_{\alpha\beta} = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}.$$

The Lie derivative w.r.t. a Killing field annihilates the metric.

ζμ is a Killing field: represents a global symmetry



From the equation of motion

$$\Gamma^0_{i,j} = 0, \qquad \Gamma^i_{0,j} = 0, \qquad \Gamma^i_{0,0} = \frac{\partial \Phi}{\partial x^i},$$

$$x^0 = ct = c\tau,$$

$$\frac{d^2 x^i}{d\tau^2} = -\frac{\partial \Phi}{\partial x^i}.$$

The metric represents the potential of the gravitational field
The connection the strength of of the field.

Summing up

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0,$$

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}.$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}).$$

$$x^0 = ct = c\tau,$$

$$\frac{d^2x^i}{d\tau^2} = -\frac{\partial\Phi}{\partial x^i}.$$

$$\Gamma^0_{i,j} = 0, \qquad \Gamma^i_{0,j} = 0, \qquad \Gamma^i_{0,0} = \frac{\partial \Phi}{\partial x^i},$$

Pseudo-Riemannian spaces

The presence of gravity is indicated by the curvature of space-time. The Riemann tensor, or curvature tensor, provides a measure of this curvature:

$$R^{\sigma}_{\mu\nu\lambda} = \Gamma^{\sigma}_{\mu\lambda,\nu} - \Gamma^{\sigma}_{\mu\nu,\lambda} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu}.$$

The form of the Riemann tensor for an affine-connected manifold can be obtained through a coordinate transformation that makes the affine connection vanish everywhere, i.e.

$$\overline{\Gamma}^{\sigma}_{\mu\nu}(\overline{x}) = 0, \quad \forall \overline{x},\ \sigma,\ \mu,\ \nu.$$

The coordinate system $\overline{x}^{\mu}$ exists if

$$\Gamma^{\sigma}_{\mu\lambda,\nu} - \Gamma^{\sigma}_{\mu\nu,\lambda} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} = 0$$

The Ricci tensor is defined as

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu} = R^{\sigma}_{\mu\sigma\nu}.$$

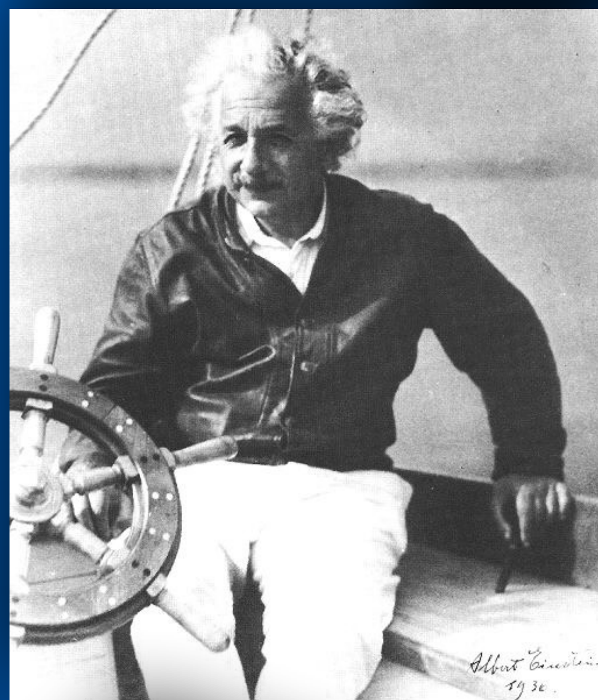
Finally, the Ricci scalar is

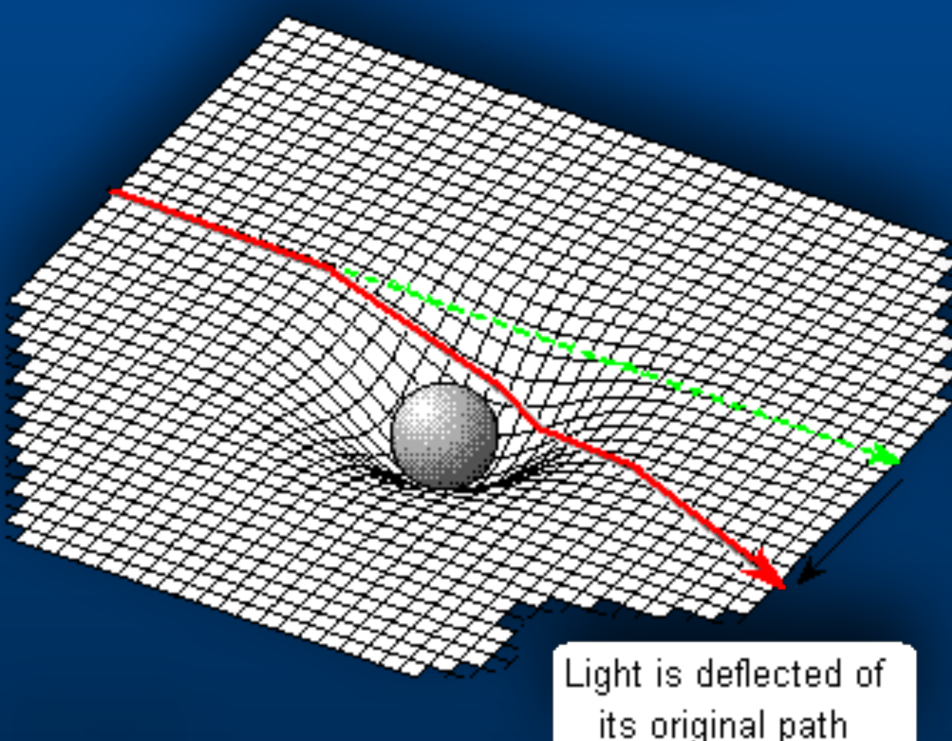
$$R = g^{\mu\nu} R_{\mu\nu}.$$

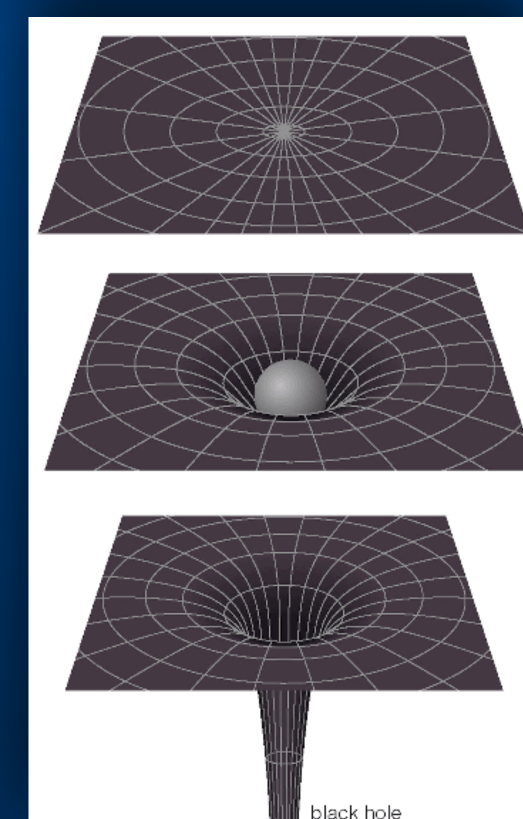
The problem is how curvature relates to the physical properties of material things. The most general aspect of things is their energy-momentum representar by rank 2 tensor. For a perfect fluid:

$$T_{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} - P g_{\mu\nu},$$

Towards General Relativity







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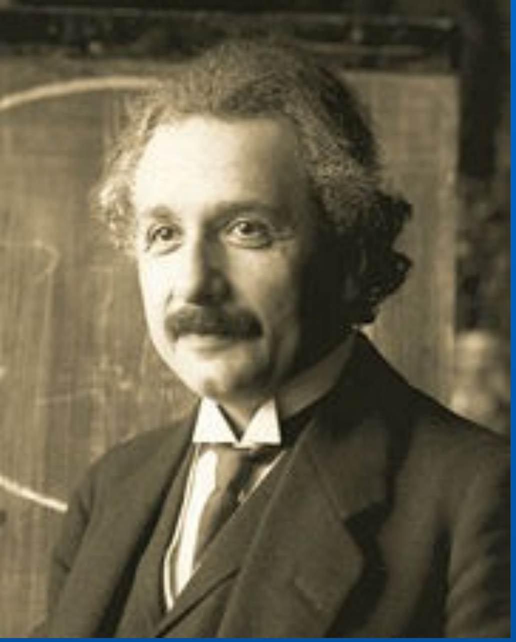
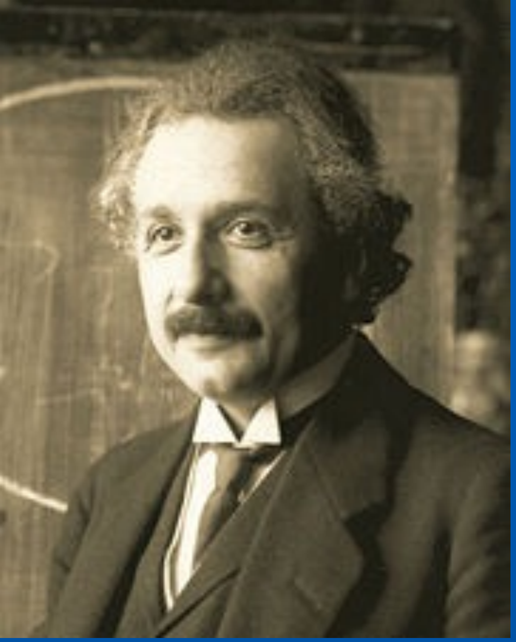


Einstein field equations

The field equations of General Relativity specify how the energy-momentum tensor is related to the curvature.

$$K_{\mu\nu} = \kappa T_{\mu\nu},$$

(i) the Newtonian limit $\nabla^2 \Phi = 4\pi G\rho$ suggests that it should contain terms no higher than linear in the second order derivatives of the metric tensor;
(ii) since $T_{\mu\nu}$ is symmetric then $K_{\mu\nu}$ must be symmetric as well.

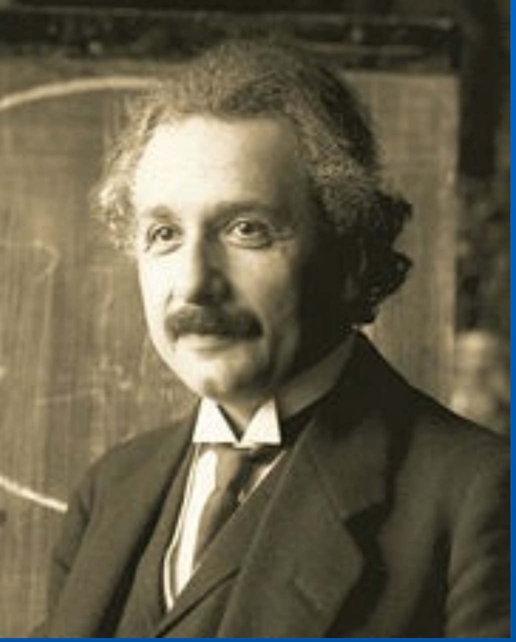


Einstein field equations

The field equations of General Relativity specify how the energy-momentum tensor is related to the curvature.

$$K_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu} + \lambda g_{\mu\nu},$$

$$K_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu}.$$



Einstein field equations

The conservation of energy-momentum requires that $T^{\mu\nu}_{\ ;\mu} = 0.$

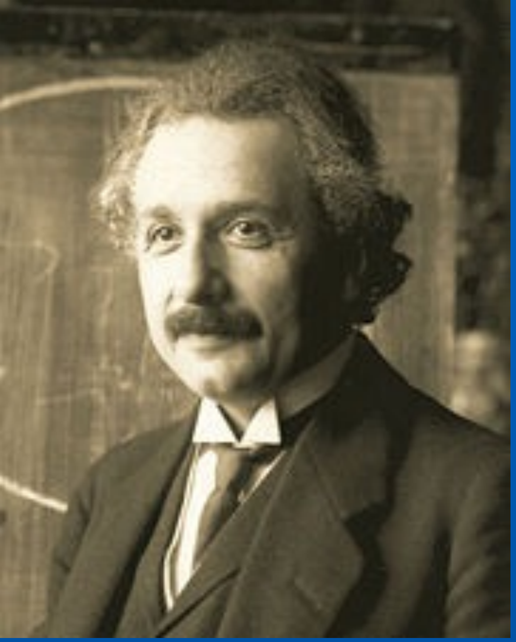
$$\left(aR^{\mu\nu} + bRg^{\mu\nu}\right)_{;\mu} = 0.$$

Bianchi's identities

$$\left(R^{\mu\nu} - \frac{1}{2}R\,g^{\mu\nu}\right)_{;\mu} = 0.$$

$\Rightarrow$ $b = -a/2$ and $a = 1.$

$$\left(R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}\right) = \kappa T_{\mu\nu}.$$

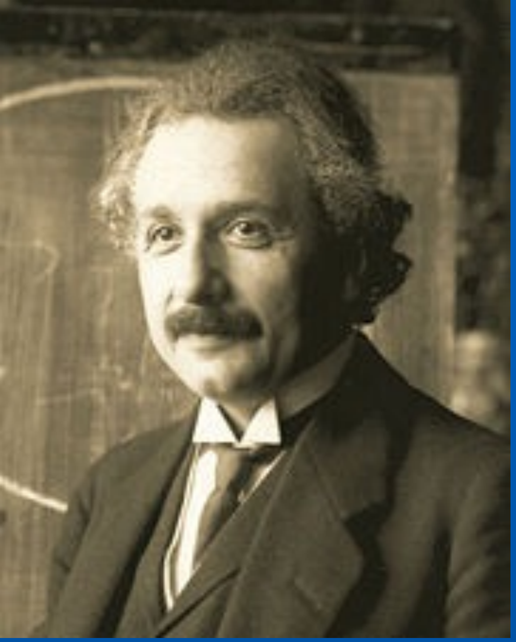


Einstein field equations

Comparing with the weak field limit: $\kappa = -8\pi G/c^4.$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}.$$

This is a set of ten non-linear partial differential equations for the metric coefficients. In Newtonian gravity, otherwise, there is only one gravitational field equation. General Relativity involves numerous non-linear differential equations.



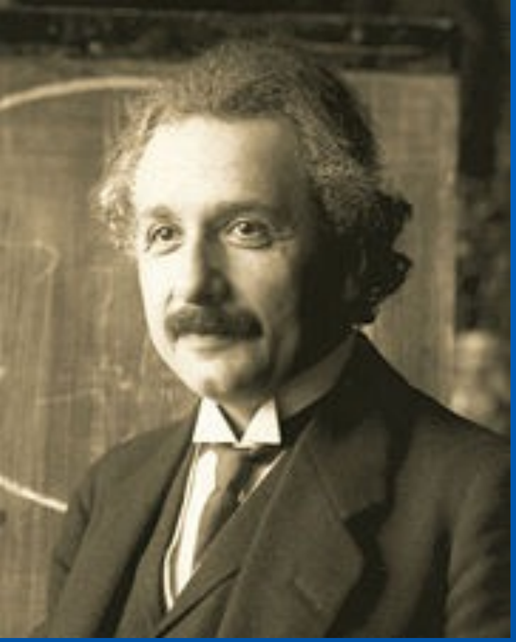
Einstein field equations

Einstein equations:

$$G_{\mu\nu} = 8\pi\, G\, T_{\mu\nu}$$

Einstein tensor (describes curvature of spacetime)

energy-momentum tensor (describes distribution of matter in the spacetime)

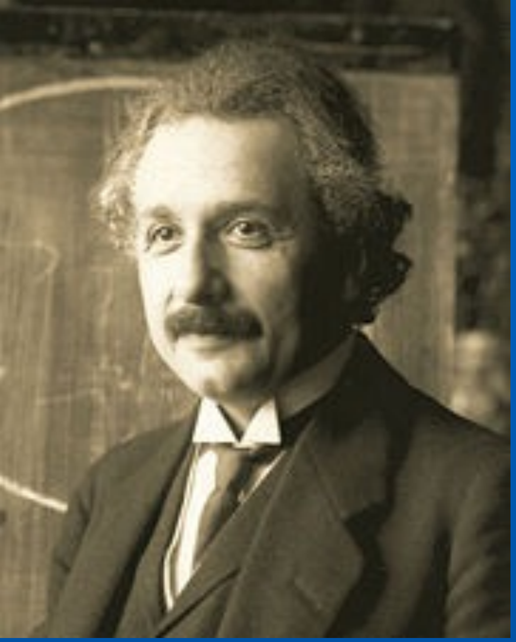


Einstein field equations

The conservation of mass-energy and momentum can be derived from the field equations:

$$T^{\mu\nu}_{\ ;\nu} = 0 \quad \text{or} \quad \nabla_\nu T^{\mu\nu} = 0.$$

Contrary to classical electrodynamics, here the field equations entail the energy-momentum conservation and the equations of motion for free particles (i.e. for particles moving in the gravitational field, treated here as a background pseudo-Riemannian space-time).



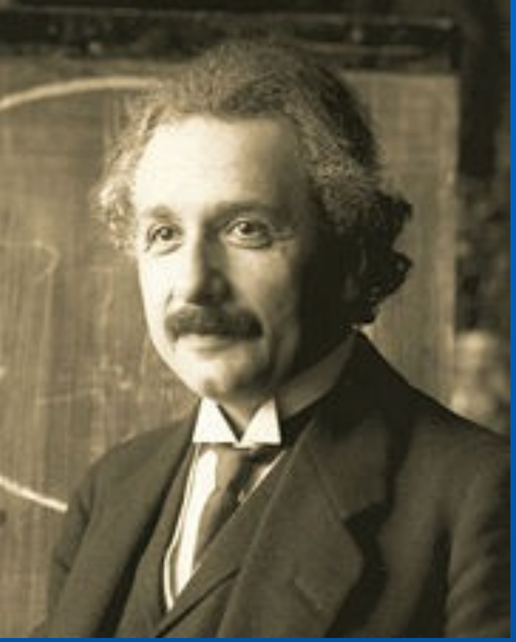
Einstein field equations

$$R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R = -\frac{8\pi G}{c^4} T^{\mu}_{\nu}.$$

$$R = -\frac{16\pi G}{c^4} T,$$

$$T = T^{\mu}_{\mu}.$$

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right).$$



Einstein field equations

In vacuum $T_{\mu\nu} = 0$

$$R_{\mu\nu} = 0,$$

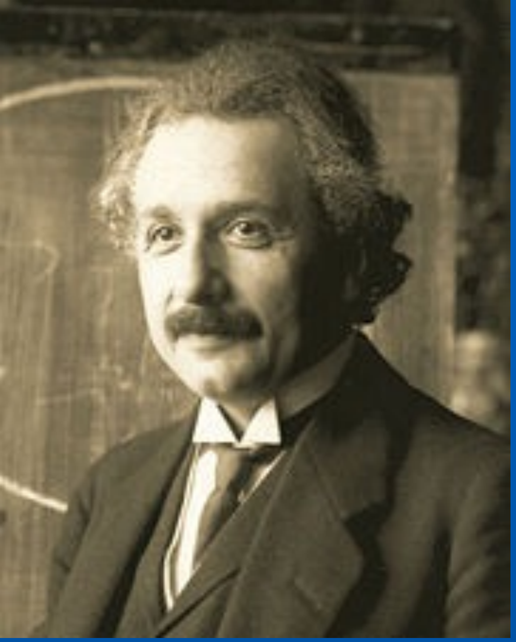
The Ricci tensor vanishes. The curvature tensor, which has 20 independent components, does not necessarily vanish. This means that a gravitational field can exist in empty space only if the dimensionality of space-time is 4 or higher. For spacetimes with lower dimensionality, the curvature tensor vanishes if $T_{\mu\nu} = 0$

Why spacetime is 4D?

$$R_{\mu\nu} = 0.$$

No. of spacetime dimensions	2	3	4
No. of field equations	3	6	10
No. of independent components of $R_{\mu\nu\sigma\rho}$	1	6	20

Gravitation in empty space can only exists if n>3



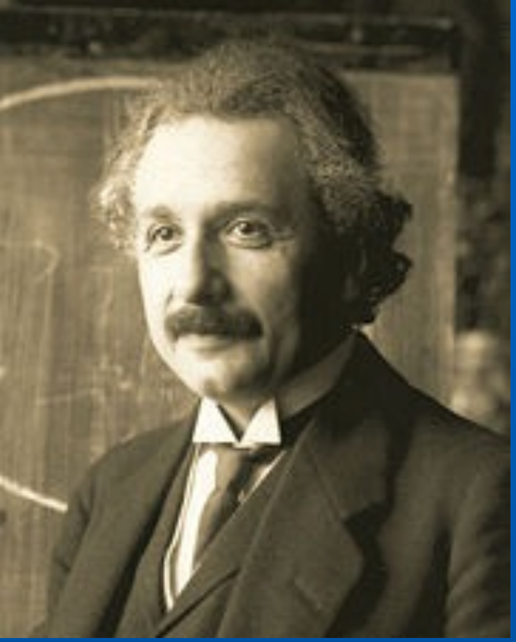
Einstein field equations with Λ

The field equations of General Relativity specify how the energy-momentum tensor is related to the curvature. They are ten non-linear differential equations for the metric coefficients.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -(8\pi G/c^4) T_{\mu\nu}.$$

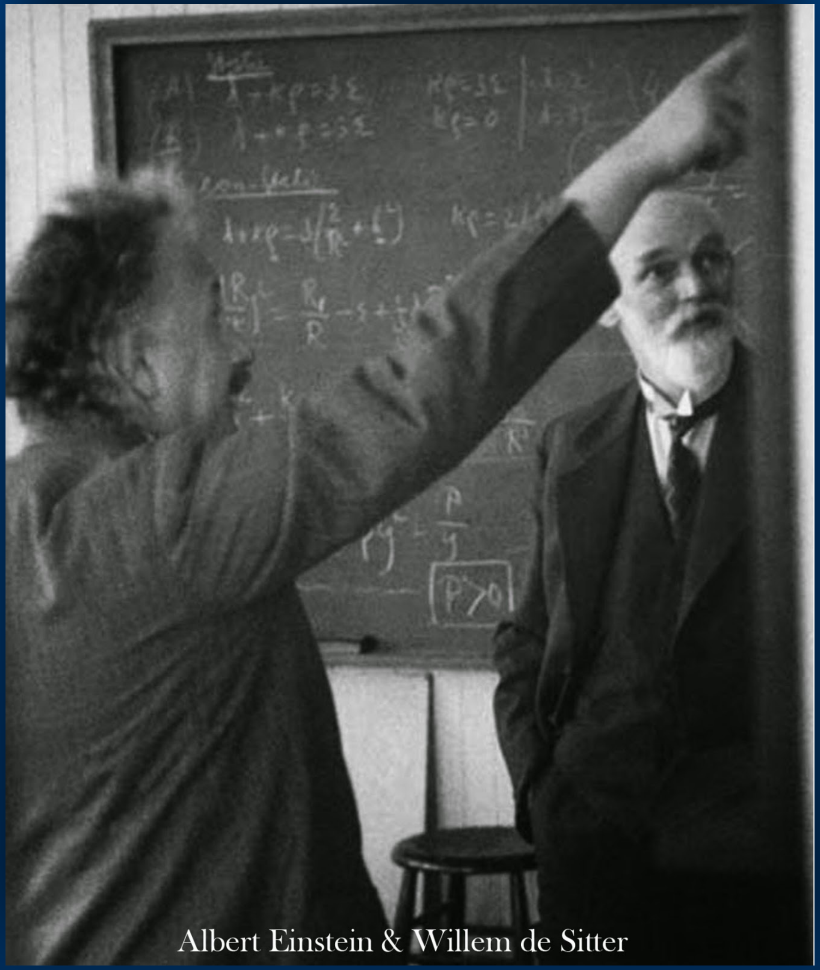
The set of equations is not unique: we can add any constant multiple of the metric tensor to the left member and still obtain a consistent set of equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -(8\pi G/c^4) T_{\mu\nu}.$$

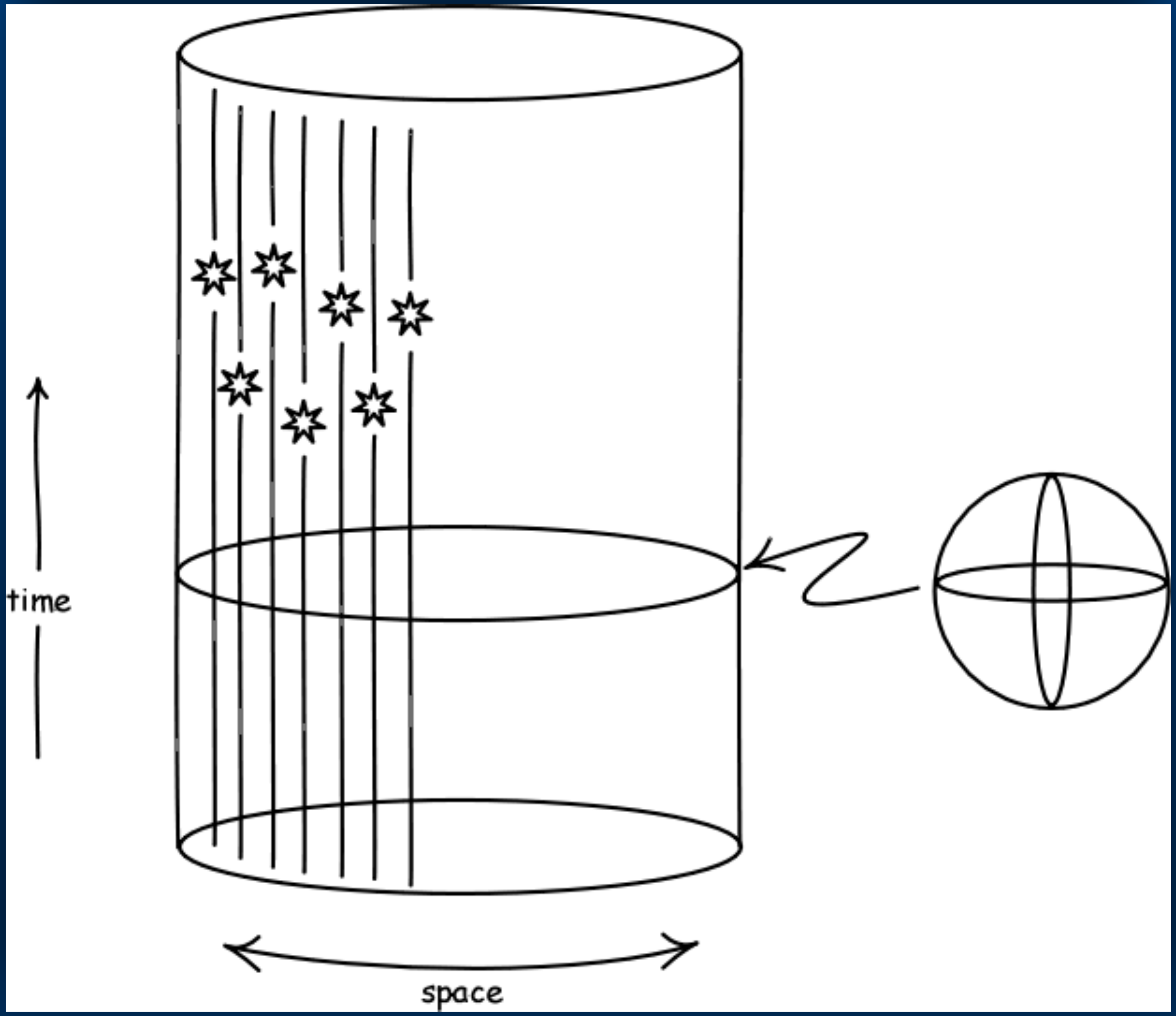


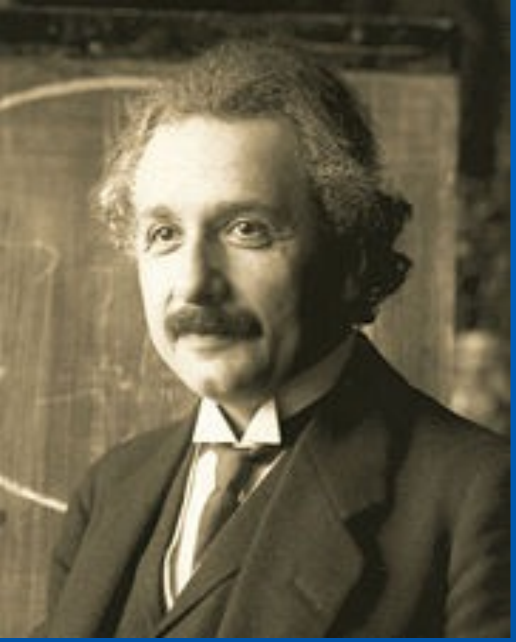
Einstein field equations with Λ

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}.$$



Albert Einstein & Willem de Sitter





Einstein field equations with Λ

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}.$$

$$\boxed{T^{\mu\nu} = (\rho + p/c^2) u^\mu u^\nu - p g^{\mu\nu}.}$$

If

$$T_{\mu\nu} = -P\, g_{\mu\nu} = \rho\, c^2 g_{\mu\nu}.$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} \left(T_{\mu\nu} + T^{\mathrm{vac}}_{\mu\nu} \right),$$

$$\rho_{\mathrm{vac}} c^2 = \frac{\Lambda c^4}{8\pi G}.$$

Hilbert's way

$$S = \int_{t_1}^{t_2} L(q^a, \dot{q}^a, t)\, dt,$$

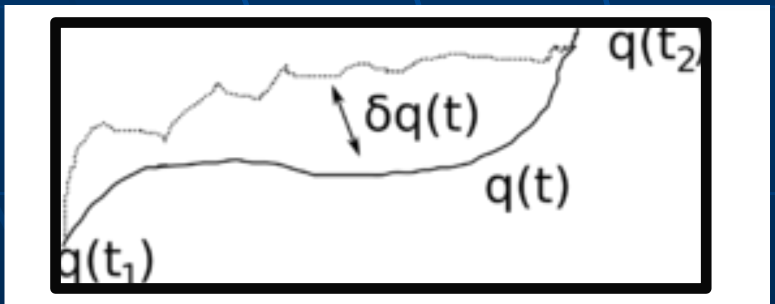
$$L = T - U = \frac{1}{2} m\, g_{ab}\, \dot{q}^a\, \dot{q}^b - U,$$

$$ds^2 = g_{ab}\, dq^a\, dq^b.$$

$$q^a(t) \to q'^a(t) = q^a(t) + \delta q^a(t),$$

$$\delta S = 0$$

$$\delta q^a(t) = 0$$



$$\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right) = 0, \quad a = 1, 2, \ldots, n.$$

Hilbert's way

$$S = \int_{\mathcal{R}} \mathcal{L}(\Phi^a,\, \partial_\mu \Phi^a,\, \partial_\mu \partial_\nu \Phi^a, \ldots)\, d^4x,$$

$$d^4x = dx^0 dx^1 dx^2 dx^3$$

$$\mathcal{L} = L\sqrt{-g}.$$

$$S = \int_{\mathcal{R}} L\sqrt{-g}\, d^4x,$$

Hilbert's way

$$\Phi^a(x) \to \Phi'^a(x) = \Phi^a + \delta\Phi^a(x).$$

$$\partial_\mu \Phi^a \to \partial_\mu \Phi'^a = \partial_\mu \Phi^a + \partial_\mu\left(\delta\Phi^a\right).$$

$$S \to S + \delta S$$

$$\delta S = \int_{\mathcal{R}} \delta\mathcal{L}\, d^4x = \int_{\mathcal{R}} \left[\frac{\partial \mathcal{L}}{\partial \Phi^a} \delta\Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \delta\left(\partial_\mu \Phi^a\right) \right] d^4x.$$

$$\frac{\delta \mathcal{L}}{\delta \Phi^a} = \frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right] = 0.$$

$$\mathcal{L} = R\sqrt{-g}.$$

$$S_{\mathrm{EH}} = \int_{\mathcal{R}} R\sqrt{-g}\, d^4x.$$

Einstein-Hilbert action

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu},$$

$$\delta S_{\mathrm{EH}} = \int_{\mathcal{R}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \sqrt{-g}\, d^4x.$$

$$\delta S_{\mathrm{EH}} = \int_{\mathcal{R}} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} \sqrt{-g}\, d^4x.$$

$$\delta S_{\mathrm{EH}} = 0$$

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$

If there are non-gravitational fields present the action will have and additional component:

$$S = \frac{1}{2\kappa} S_{\mathrm{EH}} + S_{\mathrm{M}} = \int_{\mathcal{R}} \left(\frac{1}{2\kappa} \mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{M}} \right) d^4x,$$

$$\frac{1}{2\kappa}\frac{\delta \mathcal{L}_{\mathrm{EH}}}{\delta g^{\mu\nu}} + \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}} = 0.$$

Since $\delta S_{\mathrm{EH}} = 0$,

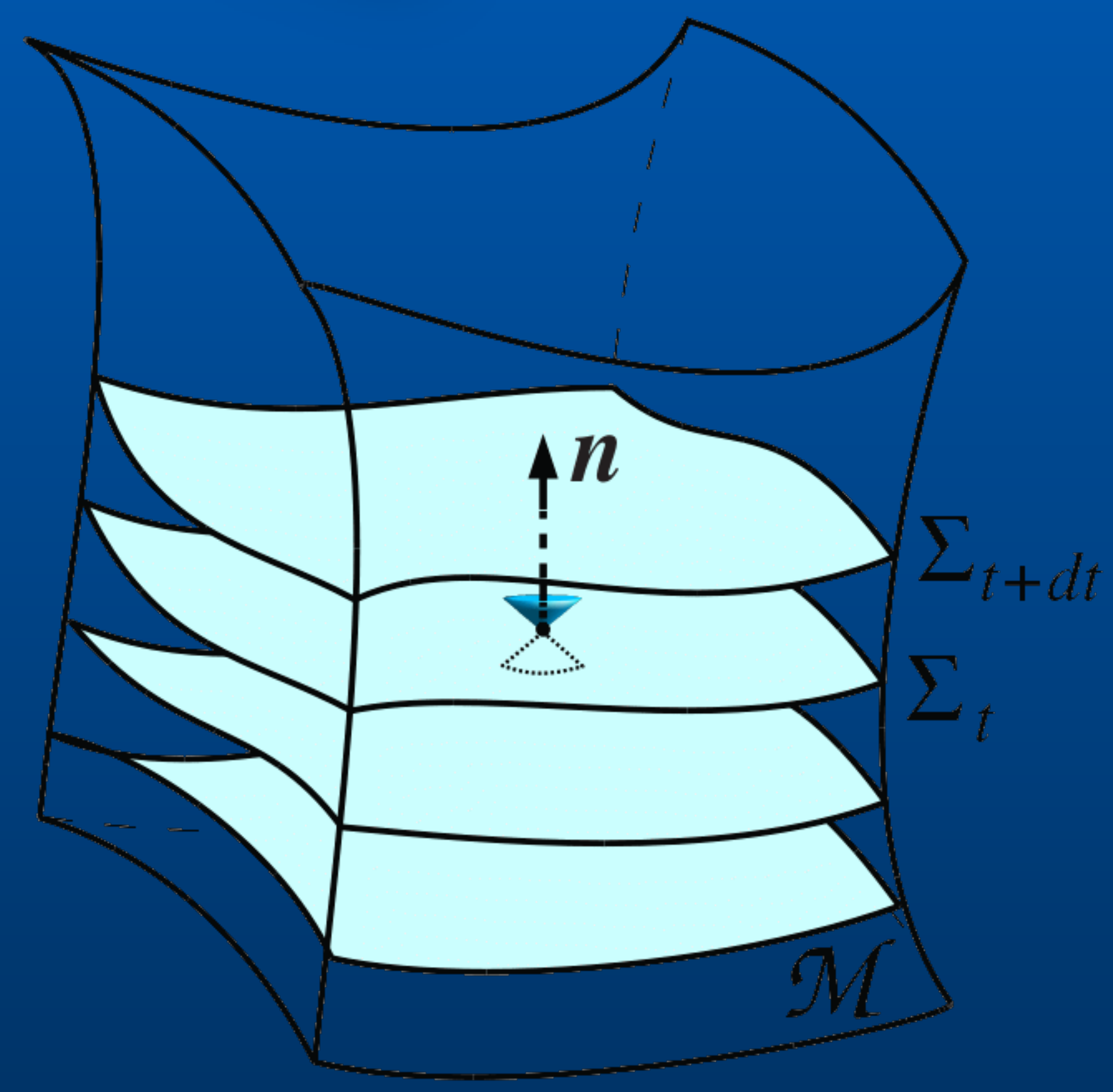
$$\frac{\delta \mathcal{L}_{\mathrm{EH}}}{\delta g^{\mu\nu}} = \sqrt{-g}\, G_{\mu\nu}.$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g^{\mu\nu}},$$

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}.$$

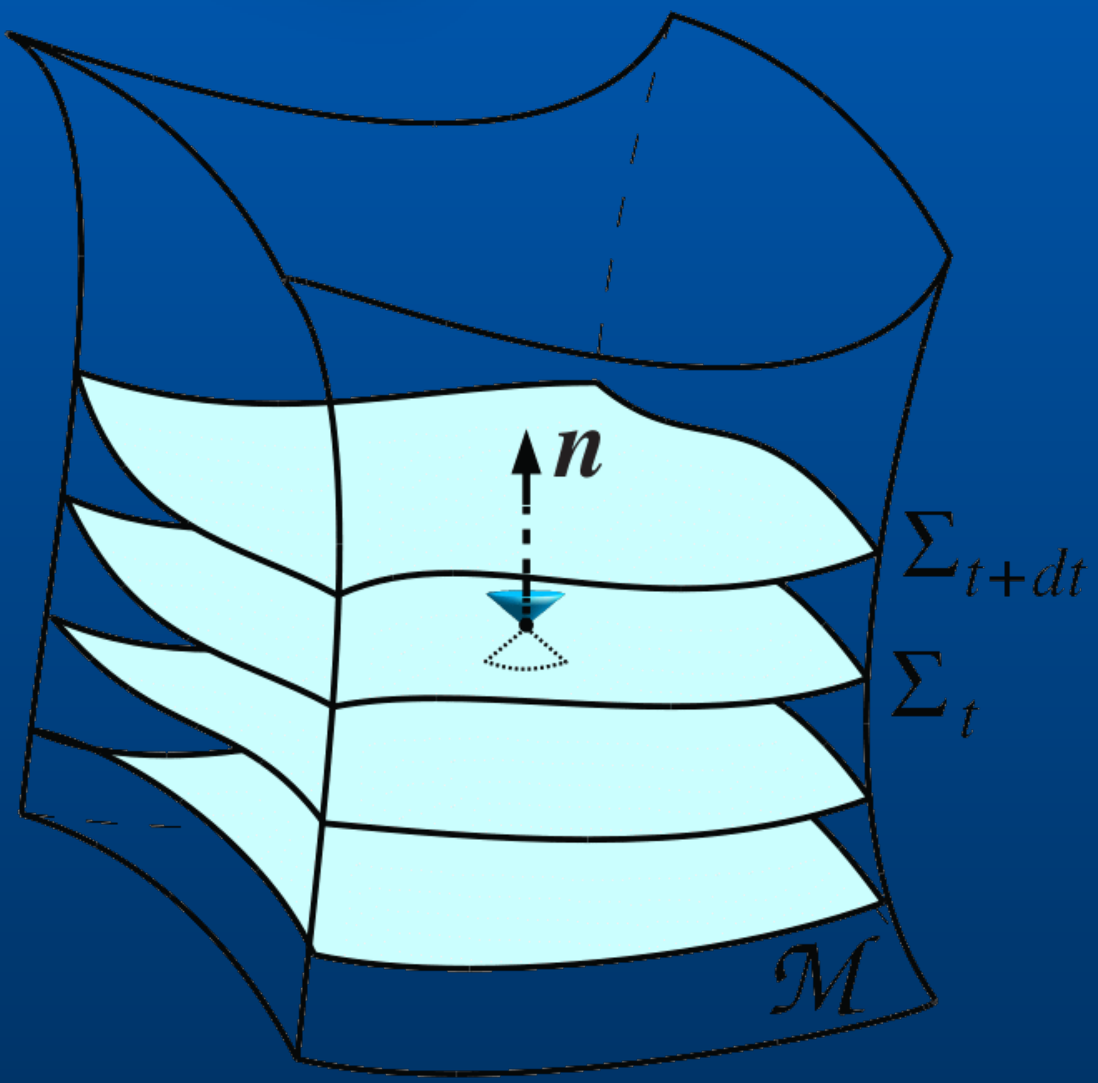
The Cauchy problem in GR

The Cauchy problem is the problem of given the metric tensor and its derivatives at one time x_0, then construct the metric which corresponds to a vacuum spacetime for all future time.



The Cauchy problem in GR

Let us prescribe initial data $g_{\mu\nu}$ and $g_{\mu\nu,0}$ on S defined by $x_0/c=t$. The dynamical equations are the six equations defined by



$$G^{i,j} = -\frac{8\pi G}{c^4}T^{ij}.$$

The Cauchy problem in GR

When these equations are solved for the 10 second derivatives $\partial^2 g_{\mu\nu}/\partial(x^0)^2$, there appears a fourfold ambiguity, i.e. four derivatives are left indeterminate. In order to completely fix the metric it is necessary to impose four additional conditions. These conditions are usually imposed upon the affine connection:

$$\Gamma^\mu \equiv g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0.$$

The Cauchy problem in GR

The condition $\Gamma^\mu = 0$ implies $\Box^2 x^\mu = 0$, so the coordinates are known as harmonic. With such conditions it can be shown the existence, uniqueness and stability of the solutions.

Conservation laws

Taking the covariant derivative to both sides of Einstein's equations and using Bianchi identities we get

$$\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)_{;\mu}=0, \quad \Longrightarrow \quad T^{\mu\nu}_{\ ;\mu}=0.$$

For a perfect fluid

$$\boxed{T^{\mu\nu}=(\rho+p/c^2)u^\mu u^\nu-pg^{\mu\nu}.} \quad \Longrightarrow$$

$$\boxed{\partial_\mu(\rho u^\mu)+(p/c^2)\partial_\mu u^\mu=0.}$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{u})=0,$$

Continuity

$$\boxed{(\rho+p/c^2)(\partial_\mu u^\nu)u^\mu=(\eta^{\mu\nu}-u^\mu u^\nu/c^2)\partial_\mu p.}$$

$$\rho\left(\frac{\partial}{\partial t}+\vec{u}\cdot\vec{\nabla}\right)\vec{u}=-\vec{\nabla}p,$$

Euler

Energy-momentum of gravitation

Because of the Equivalence Principle, it is always possible to choose a coordinate system where the gravitational field locally vanishes. Hence, its local energy is zero.

We can then define a quasi-tensor for the energy-momentum of gravity. Quasi-tensors are objects that under *global* linear transformations behave like tensors.

$$\Theta^{\mu\nu}{}_{,\nu} = 0.$$

$$\Theta^{\mu\nu} = \sqrt{-g}\left(T^{\mu\nu} + t^{\mu\nu}\right) = \Lambda^{\mu\nu\alpha}{}_{,\alpha}\,.$$

Since $t_{\mu\nu}$ can be interpreted as the contribution of gravitation to the quasi-tensor $\Theta_{\mu\nu}$, we can expect that it should be expressed in geometric terms only, i.e. as a function of the affine connection and the metric. Landau and Lifshitz (1962) found an expression for $t_{\mu\nu}$ that contains only first derivatives and is symmetric:

$$
\begin{aligned}
t^{\mu\nu} = \frac{c^4}{16\pi G}\Big[& \big(2\Gamma^{\sigma}_{\rho\eta}\Gamma^{\gamma}_{\sigma\gamma} - \Gamma^{\sigma}_{\rho\gamma}\Gamma^{\gamma}_{\eta\sigma} - \Gamma^{\sigma}_{\rho\sigma}\Gamma^{\gamma}_{\eta\gamma}\big)\big(g^{\mu\rho}g^{\nu\eta} - g^{\mu\nu}g^{\rho\eta}\big) \\
& + g^{\mu\rho}g^{\eta\sigma}\big(\Gamma^{\nu}_{\rho\gamma}\Gamma^{\gamma}_{\eta\sigma} + \Gamma^{\nu}_{\eta\sigma}\Gamma^{\gamma}_{\rho\gamma} + \Gamma^{\nu}_{\sigma\gamma}\Gamma^{\gamma}_{\rho\eta} + \Gamma^{\nu}_{\rho\eta}\Gamma^{\gamma}_{\sigma\gamma}\big) \\
& + g^{\nu\rho}g^{\eta\sigma}\big(\Gamma^{\mu}_{\rho\gamma}\Gamma^{\gamma}_{\eta\sigma} + \Gamma^{\mu}_{\eta\sigma}\Gamma^{\gamma}_{\rho\gamma} + \Gamma^{\mu}_{\sigma\gamma}\Gamma^{\gamma}_{\rho\eta} + \Gamma^{\mu}_{\rho\eta}\Gamma^{\gamma}_{\sigma\gamma}\big) \\
& + g^{\rho\eta}g^{\sigma\gamma}\big(\Gamma^{\mu}_{\rho\sigma}\Gamma^{\nu}_{\eta\gamma} - \Gamma^{\mu}_{\rho\eta}\Gamma^{\nu}_{\sigma\gamma}\big)\Big].
\end{aligned}
$$

It is possible to find in a curved spacetime a reference system such that locally $t_{\mu\nu} = 0$. Similarly, an adequate choice of curvilinear coordinates in a flat space-time can yield non-vanishing values for the components of $t_{\mu\nu}$. We infer from this that *the energy of the gravitational field is a global property in GR, not a local one*.

The Weyl tensor

The Weyl curvature tensor is the traceless component of the curvature (Riemann) tensor. In other words, it is a tensor that has the same symmetries as the Riemann tensor with the extra condition that metric contraction yields zero.

$$C_{abcd} = R_{abcd} + \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{2}{(n-1)(n-2)} R\, g_{a[c}g_{d]b},$$

The Weyl tensor

In 4 dimensions

$$C_{abcd} = R_{abcd} + \frac{1}{2}(g_{ac}R_{db} - g_{bc}R_{da} - g_{ad}R_{cb} + g_{bd}R_{ca}) + \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R.$$

The Weyl tensor

In 3 or less dimensions $C_{abcd}=0$

$$C^{a}_{bad} \equiv 0.$$

Two metrics that are *conformally related* to each other, i.e.

$$\overline{g}_{ab} = \Omega^2 g_{ab},$$

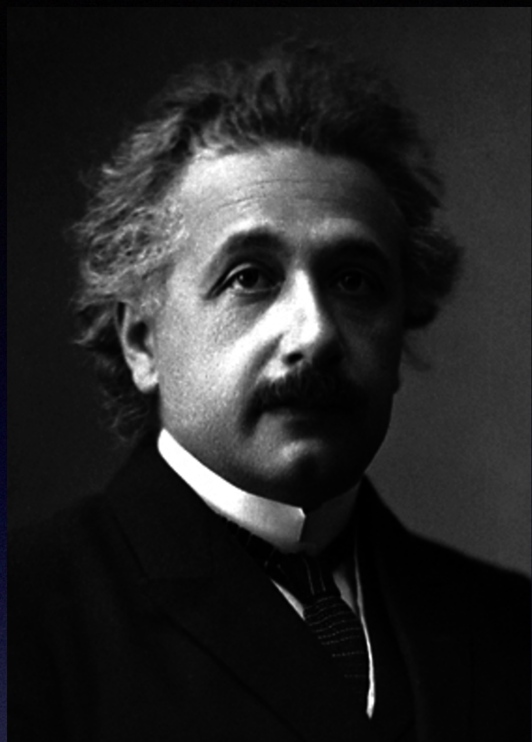
The Weyl tensor

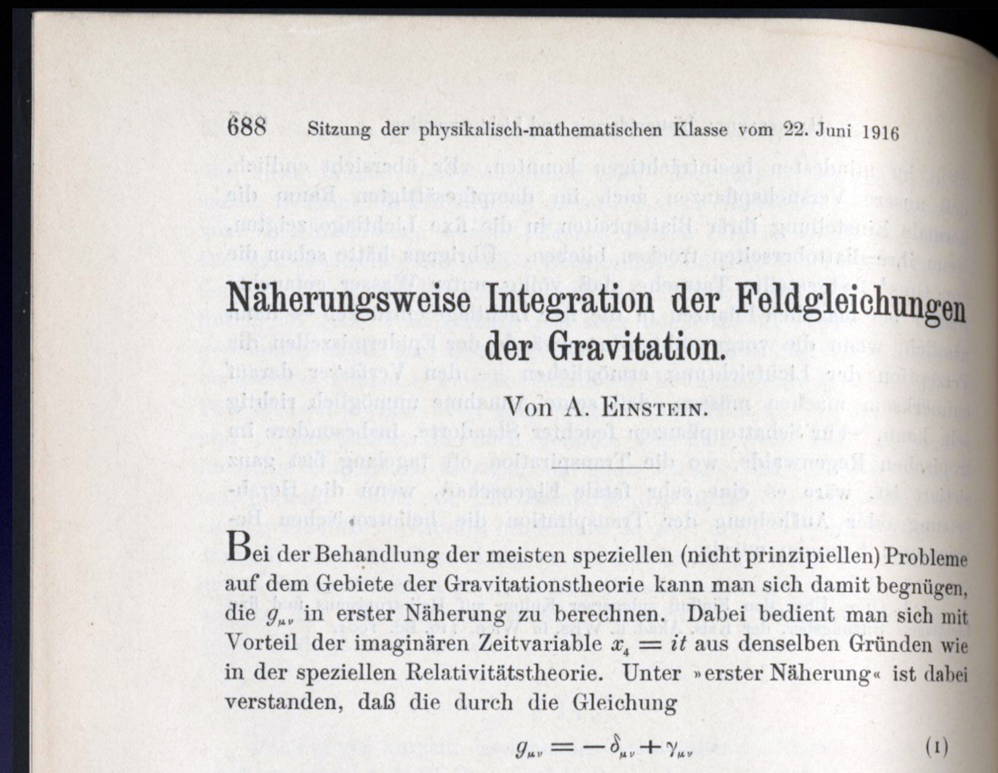
The absence of structure in space-time (i.e. spatial isotropy and hence no gravitational principal null-directions) corresponds to the absence of Weyl conformal curvature:

$$C^2 = C_{abcd}C^{abcd} = 0.$$

When clumping takes place, the structure is characterized by a non-zero Weyl curvature.

Two seminal papers





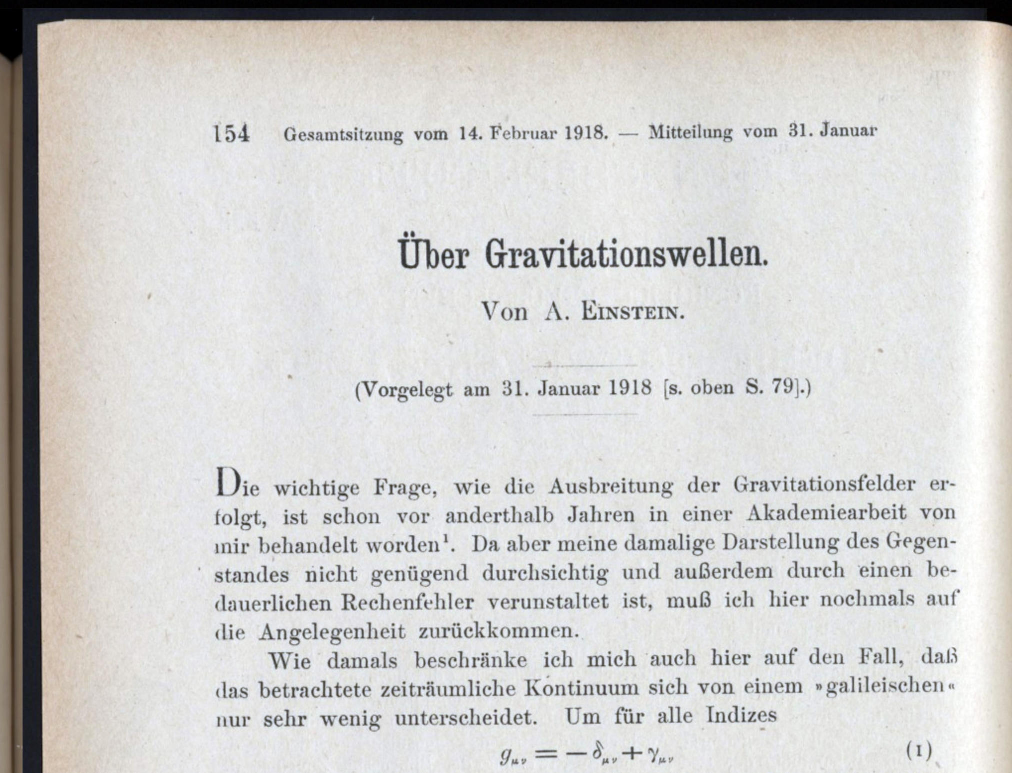
688 Sitzung der physikalisch-mathematischen Klasse vom 22. Juni 1916

Näherungsweise Integration der Feldgleichungen der Gravitation.

Von A. Einstein.

Bei der Behandlung der meisten speziellen (nicht prinzipiellen) Probleme auf dem Gebiete der Gravitationstheorie kann man sich damit begnügen, die $g_{\mu\nu}$ in erster Näherung zu berechnen. Dabei bedient man sich mit Vorteil der imaginären Zeitvariable $x_4 = it$ aus denselben Gründen wie in der speziellen Relativitätstheorie. Unter »erster Näherung« ist dabei verstanden, daß die durch die Gleichung

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \quad (1)$$



154 Gesamtsitzung vom 14. Februar 1918. — Mitteilung vom 31. Januar

Über Gravitationswellen.

Von A. Einstein.

(Vorgelegt am 31. Januar 1918 [s. oben S. 79].)

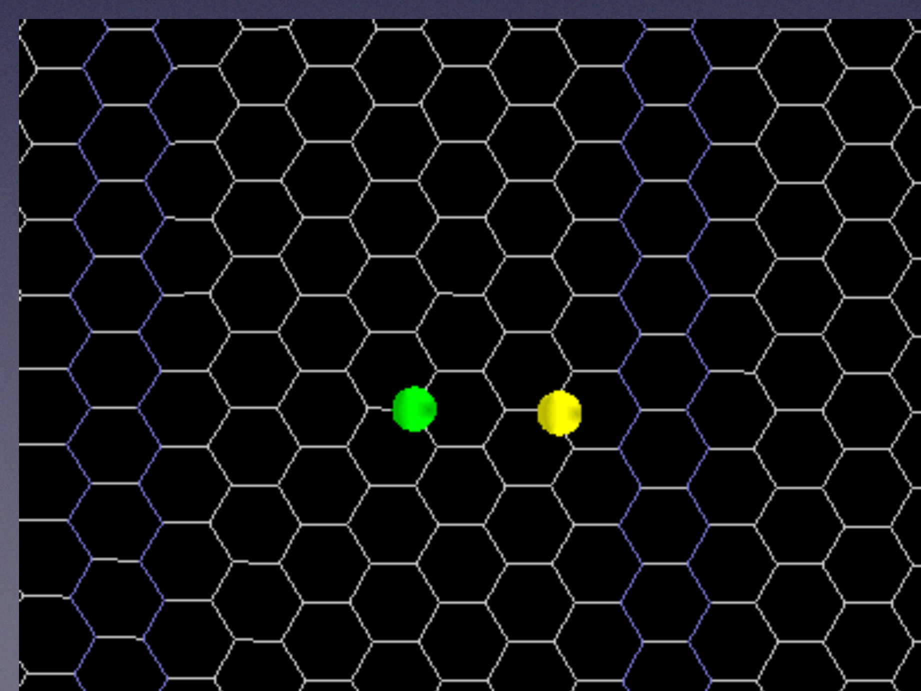
Die wichtige Frage, wie die Ausbreitung der Gravitationsfelder erfolgt, ist schon vor anderthalb Jahren in einer Akademiearbeit von mir behandelt worden[1]. Da aber meine damalige Darstellung des Gegenstandes nicht genügend durchsichtig und außerdem durch einen bedauerlichen Rechenfehler verunstaltet ist, muß ich hier nochmals auf die Angelegenheit zurückkommen.

Wie damals beschränke ich mich auch hier auf den Fall, daß das betrachtete zeiträumliche Kontinuum sich von einem »galileischen« nur sehr wenig unterscheidet. Um für alle Indizes

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} \quad (1)$$

1916

1918



GWs in linear gravity

- We consider weak gravitational fields:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2_{\mu\nu})$$

flat Minkowski metric

- The GR field equations in vacuum reduce to the standard wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) h^{\mu\nu} = \Box h^{\mu\nu} = 0$$

- Comment: GR gravity like electromagnetism has a "gauge" freedom associated with the choice of coordinate system. The above equation applies in the so-called "transverse-traceless (TT)" gauge where

$$h_{0\mu} = 0, \qquad h^{\mu}_{\mu} = 0$$

Wave solutions

- Solving the previous wave equation in weak gravity is easy. The solutions represent "plane waves":

$$h_{\mu\nu} = A_{\mu\nu} e^{ik_a x^a}$$

wave-vector

- Basic properties: $A_{\mu\nu} k^{\mu} = 0,\qquad k_a k^a = 0$

transverse waves

null vector = propagation along light rays

- Amplitude: $A^{\mu\nu} = h_+ e_+^{\mu\nu} + h_{\mathrm{x}} e_{\mathrm{x}}^{\mu\nu}$

two polarizations

$$\epsilon_+^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \epsilon_\times^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

GWs: more properties

- EM waves: at lowest order the radiation can be emitted by a dipole source (l=1). Monopolar radiation is forbidden as a result of charge conservation.

- GWs: the lowest allowed multipole is the quadrupole (l=2). The monopole is forbidden as a result of mass conservation. Similarly, dipole radiation is absent as a result of momentum conservation.

- GWs represents propagating "ripples in spacetime" or, more accurately, a propagating curvature perturbation. The perturbed curvature (Riemann tensor) is given by (in the TT gauge):

$$R^{\mathrm{TT}}_{j0k0} = -\frac{1}{2}\,\partial_t^2 h^{\mathrm{TT}}_{jk}\,, \qquad j,k = 1,2,3$$

The quadrupole formula

- Einstein (1918) derived the quadrupole formula for gravitational radiation by solving the linearized field equations with a source term:

$$\Box h^{\mu\nu}(t,\vec{x}) = -\kappa T^{\mu\nu}(t,\vec{x}) \quad \longrightarrow \quad h^{\mu\nu} = -\frac{\kappa}{4\pi}\int_V d^3x' \frac{T^{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}$$

- This solution suggests that the wave amplitude is proportional to the second time derivative of the quadrupole moment of the source:

$$h^{\mu\nu} = \frac{2}{r}\frac{G}{c^4}\ddot{Q}^{\mu\nu}_{\rm TT}(t-r/c) \qquad\qquad Q^{\mu\nu}_{\rm TT} = \int d^3x\,\rho\left(x^\mu x^\nu - \frac{1}{3}\delta^{\mu\nu}r^2\right)$$

(quadrupole moment in the "TT gauge" and at the retarded time t-r/c)

- This result is quite accurate for all sources, as long as the wavelength is much longer than the source size R.

GW luminosity

- GWs carry energy. The stress-energy carried by GWs cannot be localized within a wavelength. Instead, one can say that a certain amount of stress-energy is contained in a region of the space which extends over several wavelengths. The stress-energy tensor can be written as:

$$T_{\mu\nu}^{\mathrm{GW}} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{ij}^{\mathrm{TT}} \partial_\nu h_{\mathrm{TT}}^{ij} \rangle$$

- Using the previous quadrupole formula we obtain the GW luminosity:

$$L_{\mathrm{GW}} = \frac{dE_{\mathrm{GW}}}{dt} = \int dA\, T_{0j}^{\mathrm{GW}} \hat{n}^j \quad \longrightarrow \quad L_{\mathrm{GW}} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{Q}_{\mu\nu}^{\mathrm{TT}} \dddot{Q}_{\mathrm{TT}}^{\mu\nu} \rangle$$

Basic estimates

- Another estimate for the GW amplitude can be derived from the flux formula

$$F_{\rm GW} = \frac{L_{\rm GW}}{4\pi r^2} = \frac{c^3}{16\pi G}|\partial_t h|^2$$

- We obtain:

$$h \approx 10^{-22} \left(\frac{E_{\rm GW}}{10^{-4}\,M_\odot}\right)^{1/2} \left(\frac{1\,\text{kHz}}{f_{\rm GW}}\right) \left(\frac{\tau}{1\,\text{ms}}\right)^{-1/2} \left(\frac{15\,\text{Mpc}}{r}\right)$$

for example, this formula could describe the GW strain from a supernova explosion at the Virgo cluster during which the energy $E_{\rm GW}$ is released in GWs at a frequency of 1 kHz, and with signal duration of the order of 1 ms.

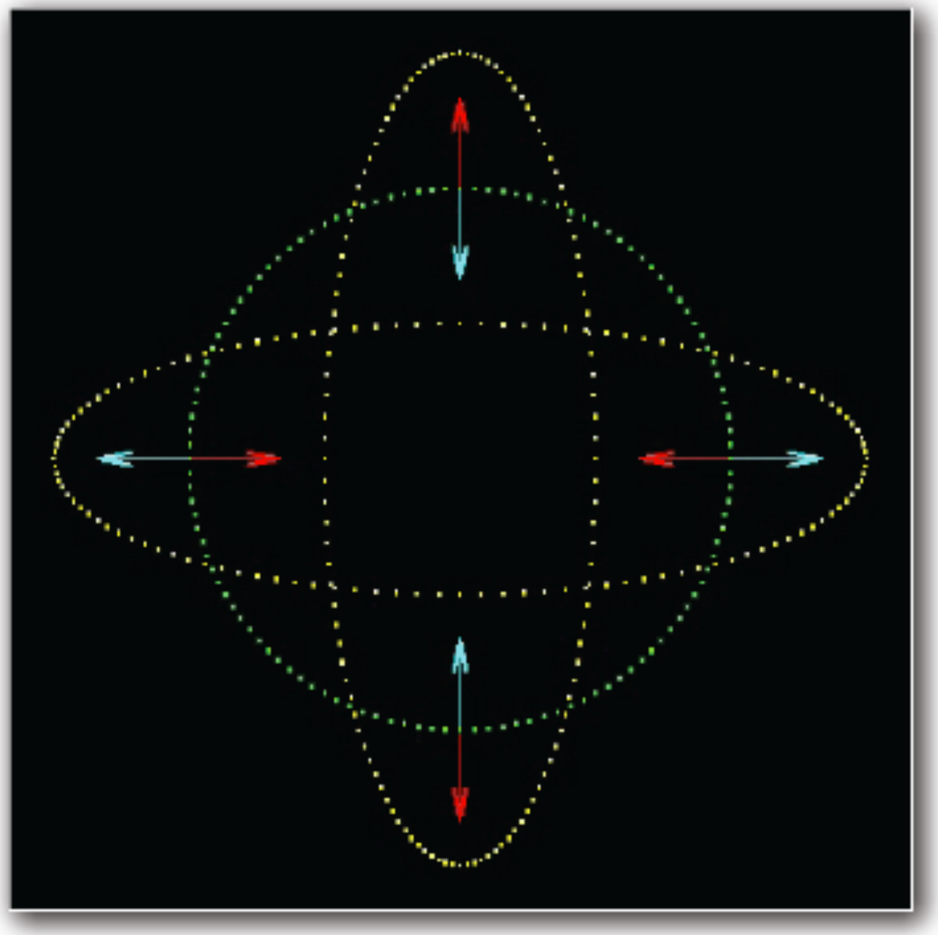
- This is why GWs are hard to detect: for a GW detector with arm length of $l = 4\,\text{km}$ we are looking for changes in the arm-length of the order of

$$\Delta l = hl = 4 \times 10^{-17}\,\text{cm} \quad !!$$

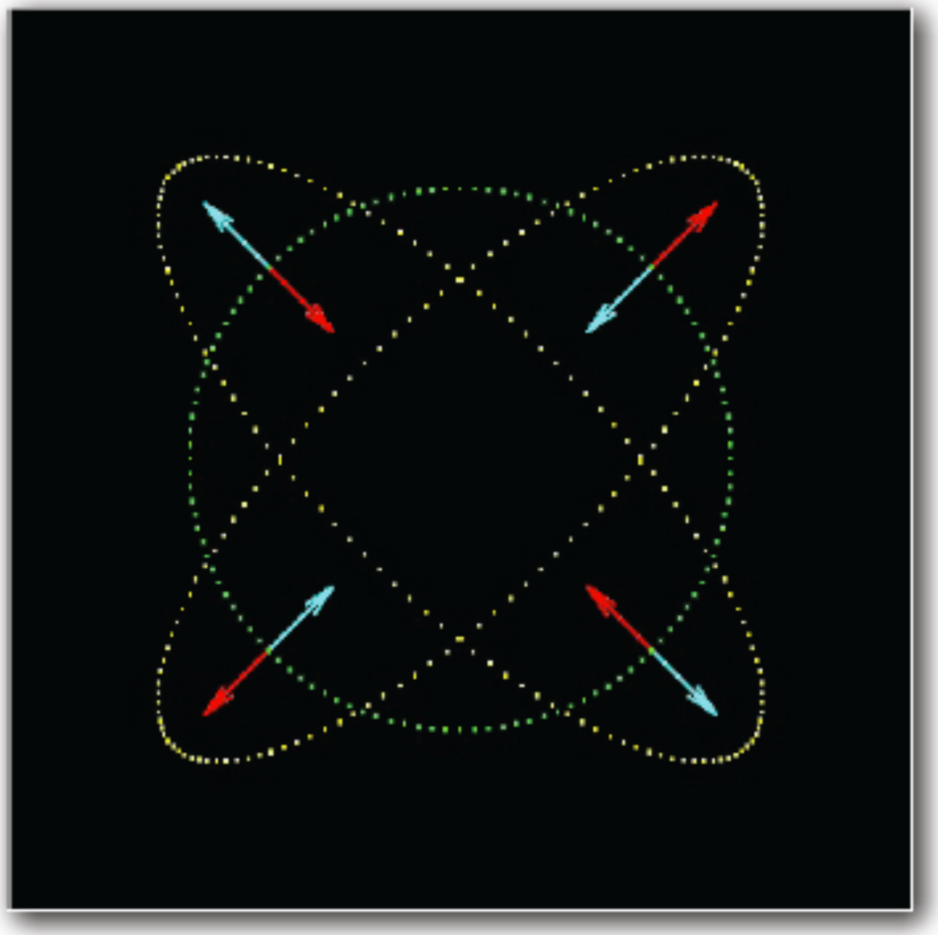
r_p=8,4184(67) × 10^{-14} cm

GWs: polarization

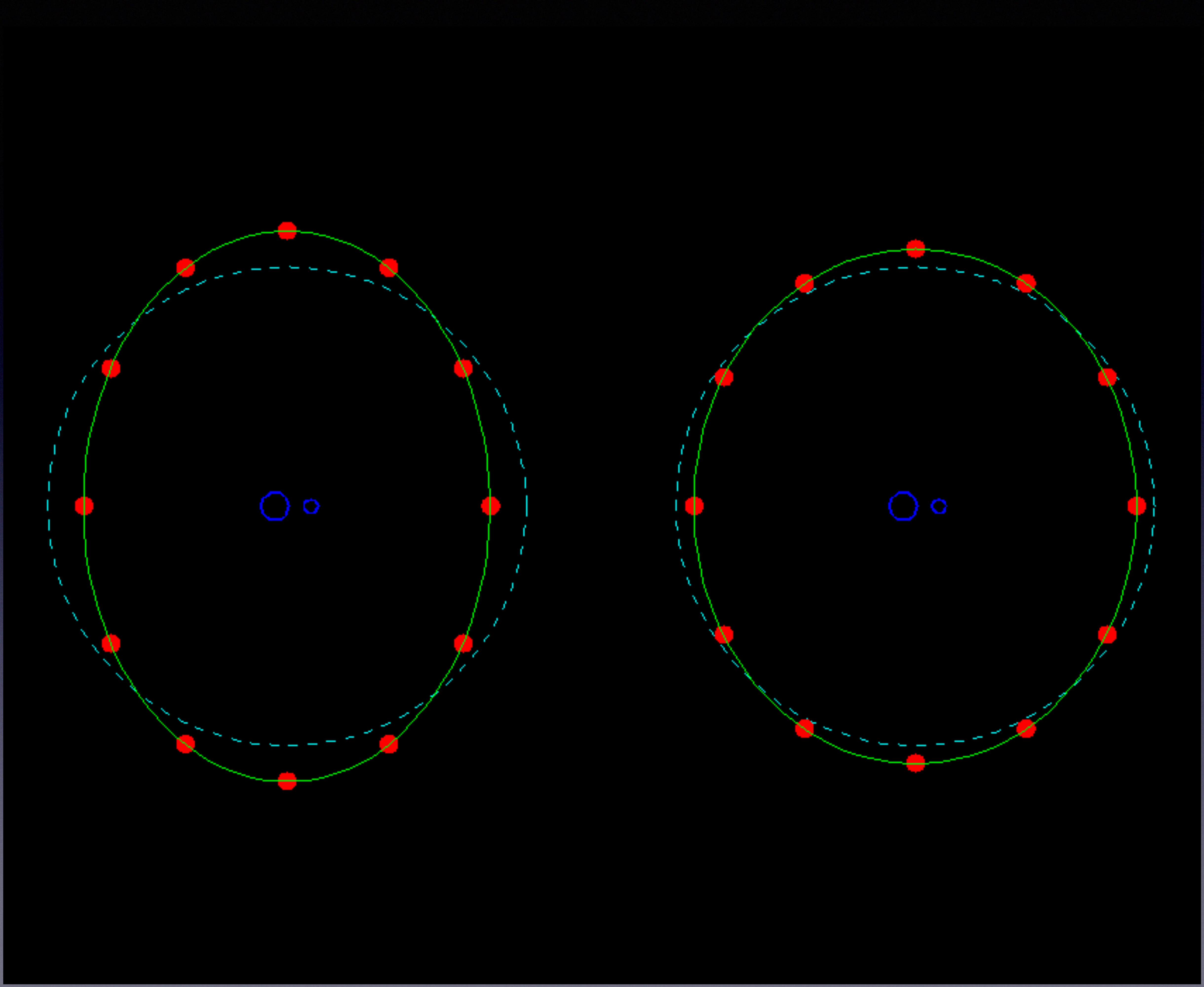
- GWs come in two polarizations:

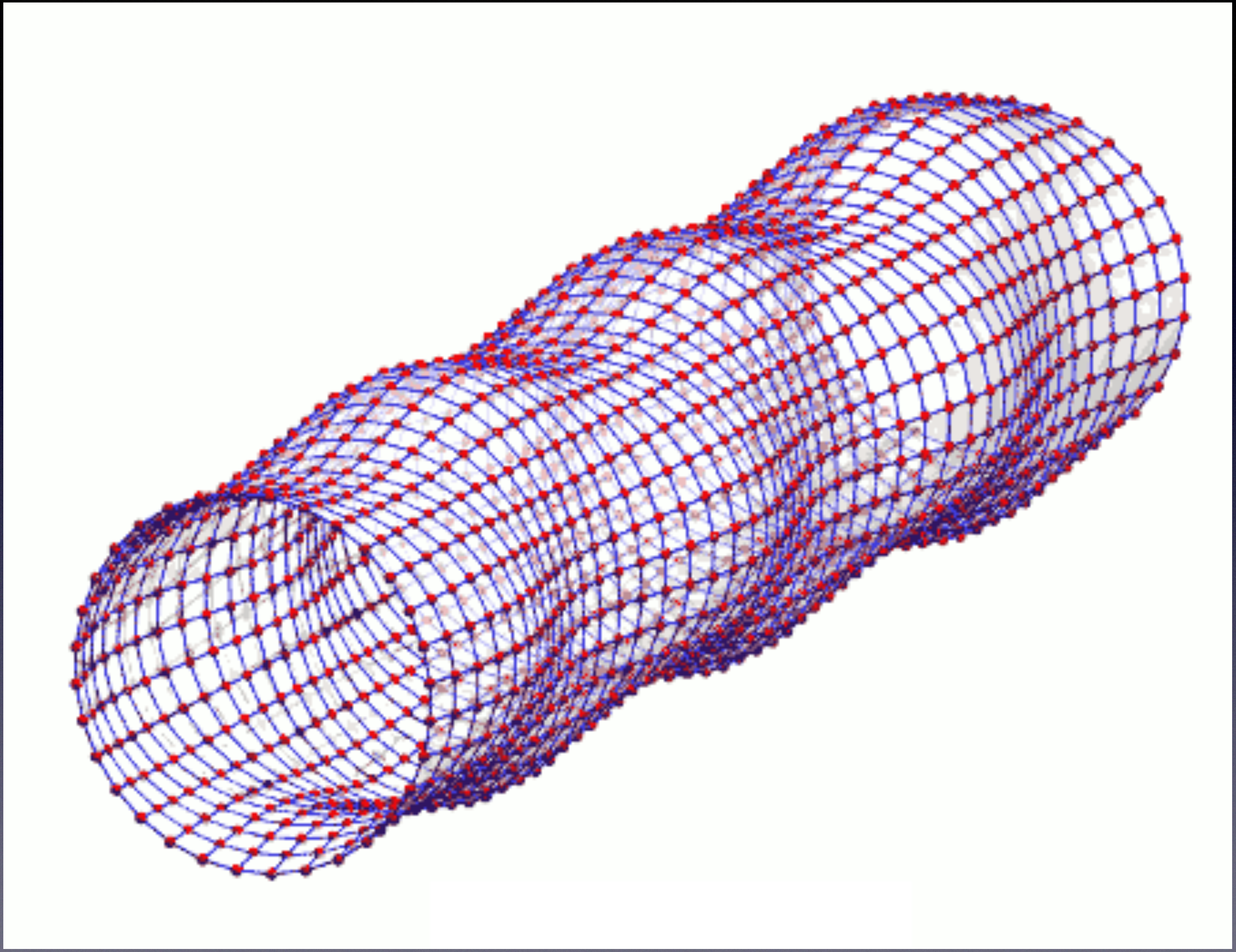


"+" polarization



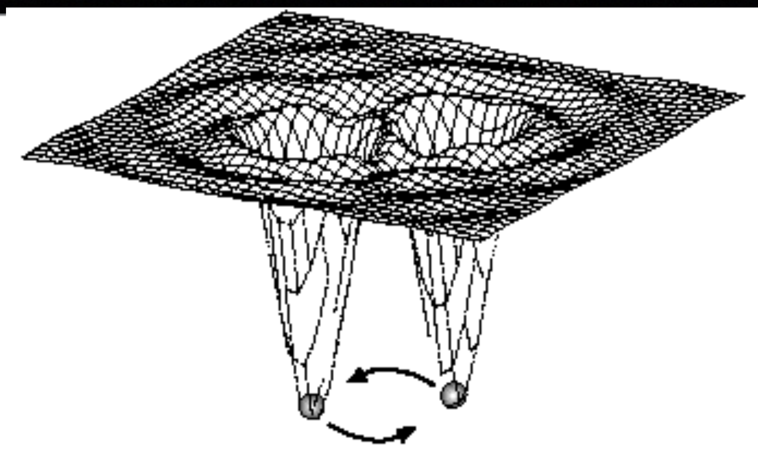
"x" polarization



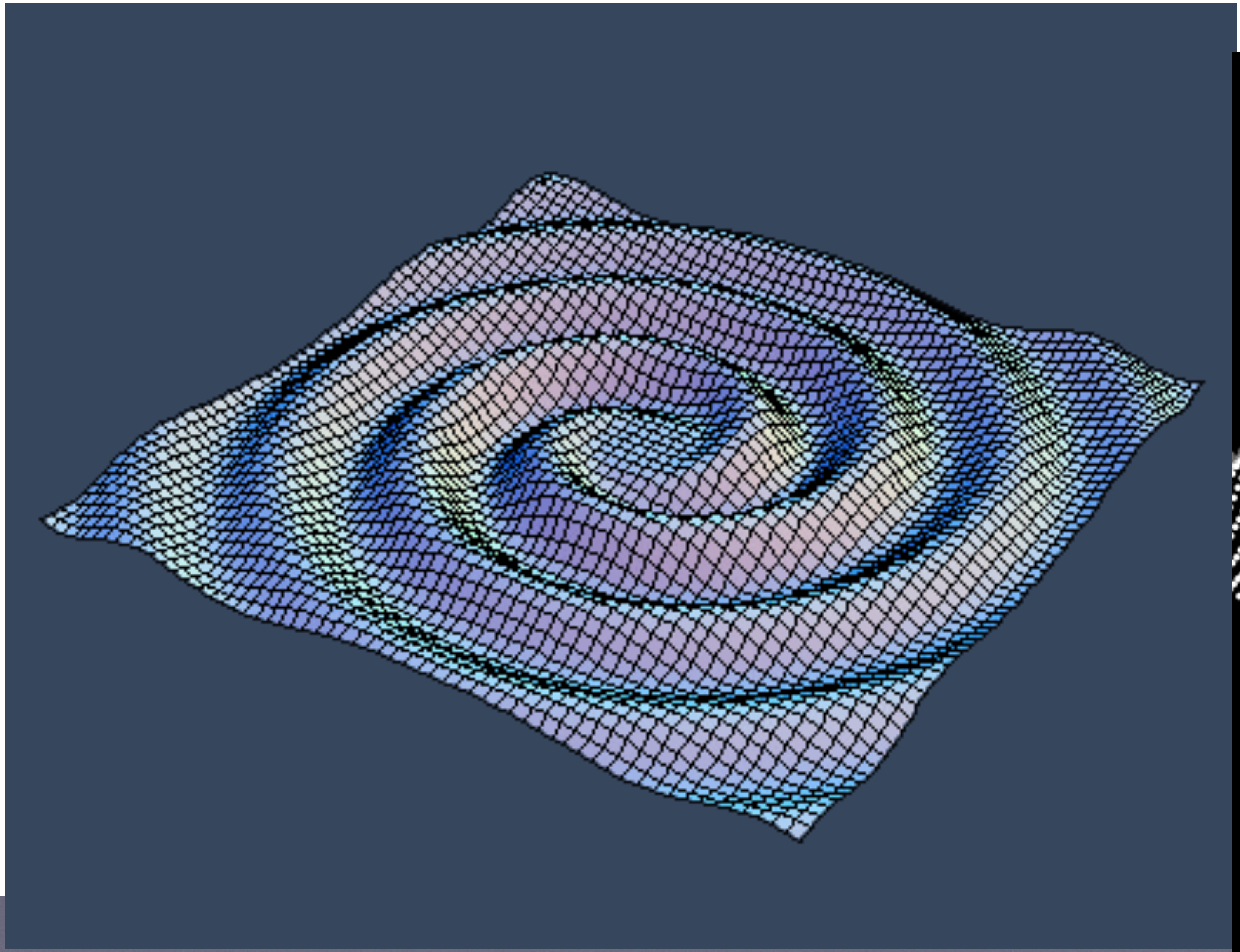


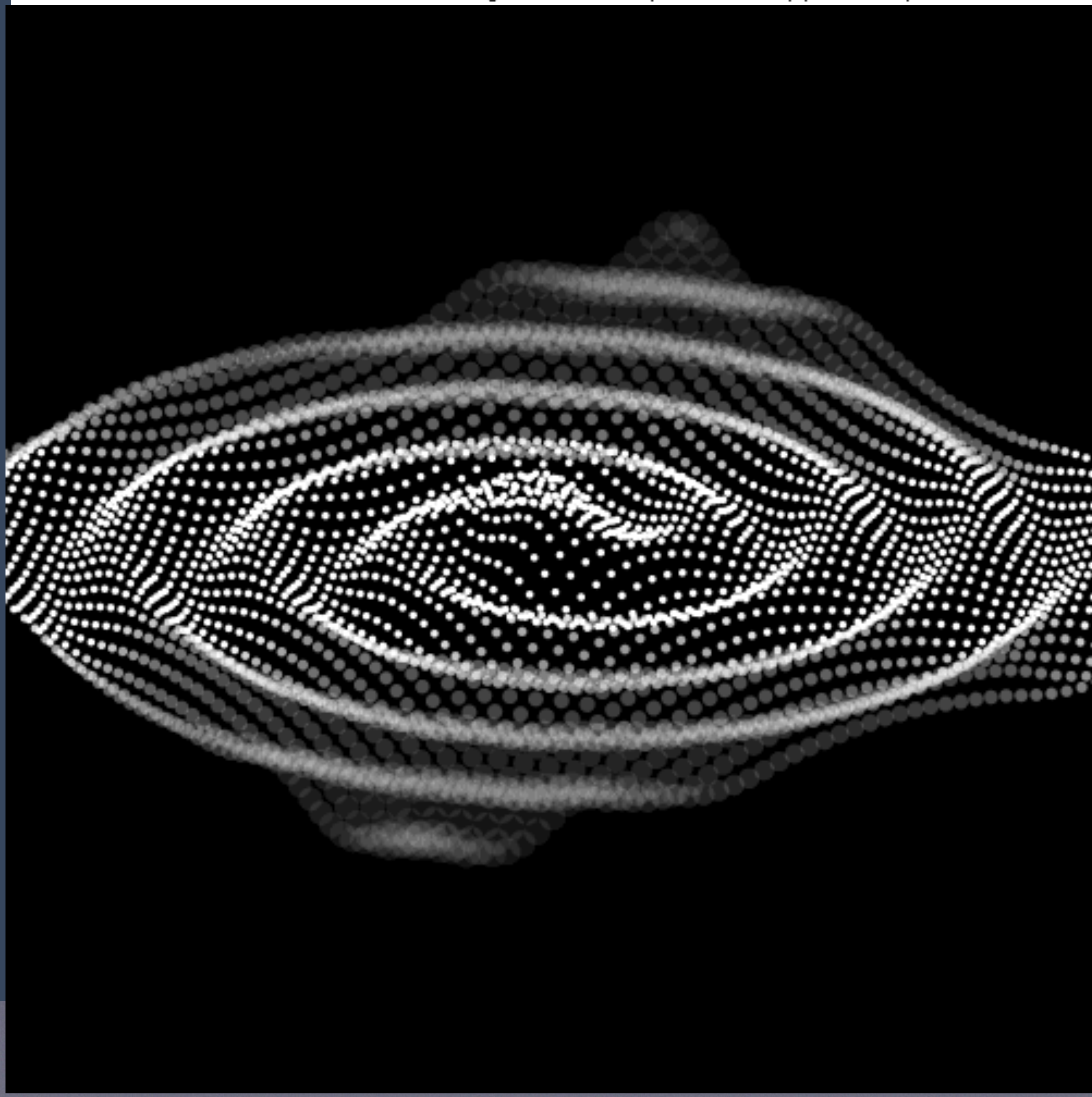
+ waves

GWs and curvature


A binary system of compact massive objects rapidly orbiting each other produces ripples in spacetime.

- As we mentioned, GWs represent a fluctuating curvature field.





GWs vs EM waves

- Similarities:

✓ Propagation with the speed of light.

✓ Amplitude decreases as ~ $1/r$.

✓ Frequency redshift (Doppler, gravitational, cosmological).

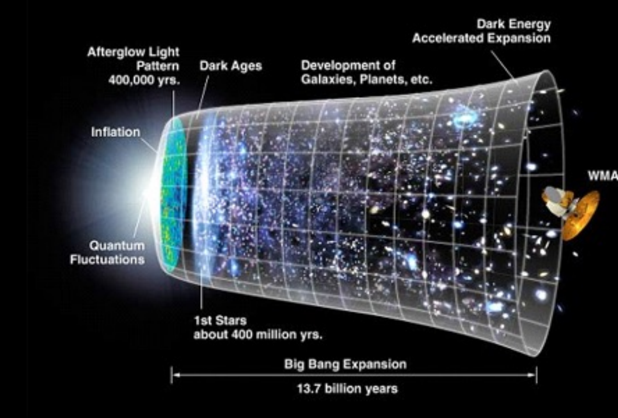
- Differences:

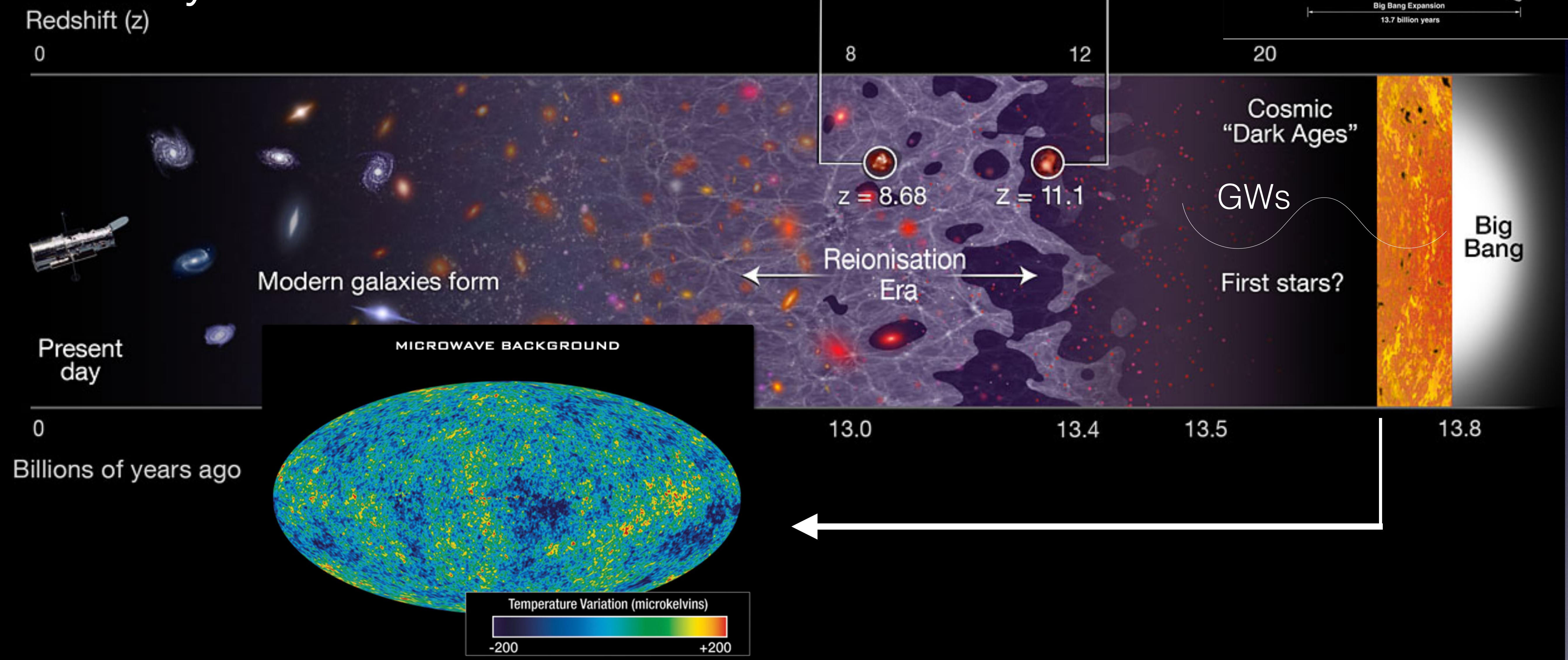
✓ GWs propagate through matter with little interaction. Hard to detect, but they carry uncontaminated information about their sources.

✓ Strong GWs are generated by bulk (coherent) motion. They require strong gravity/high velocities (compact objects like black holes and neutron star).

✓ EM waves originate from small-scale, incoherent motion of charged particles. They are subject to “environmental” contamination (interstellar absorption etc.).

GW can propagate from the inflationary period, if it existed, to the present, contrary to EM waves

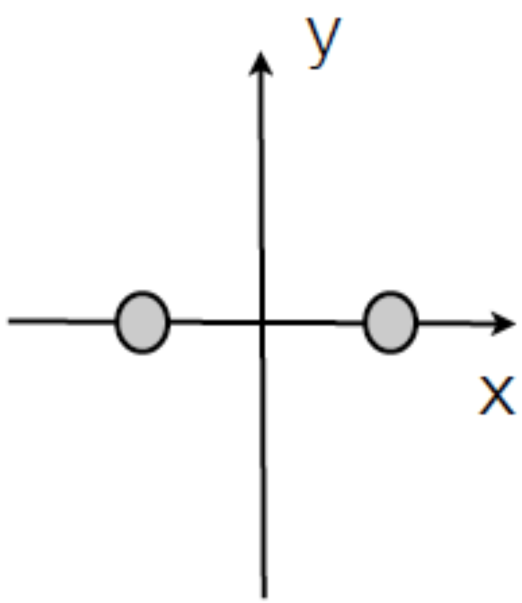




Effect on test particles

- We consider a pair of test particles on the cartesian axis Ox at distances $\pm x_0$ from the origin and we assume a GW traveling in the z-direction.

- Their distance will be given by the relation:



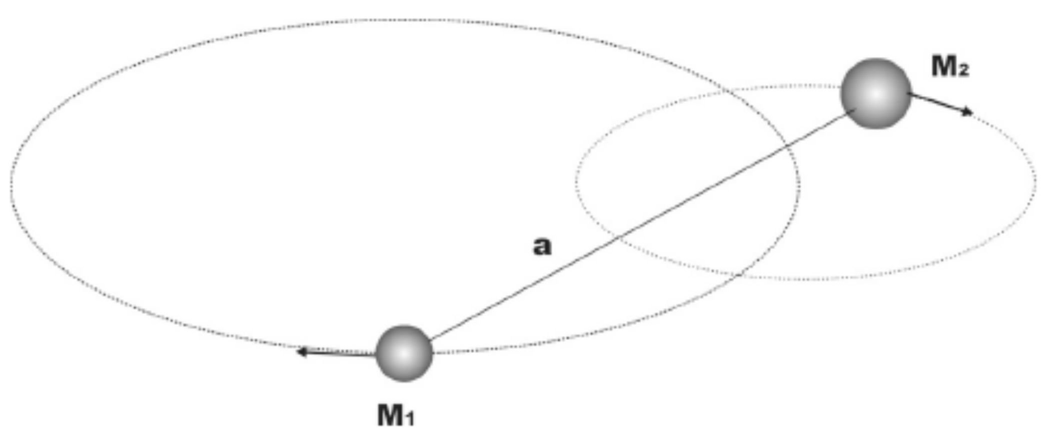
$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu = ... = -g_{11} dx^2 =$$

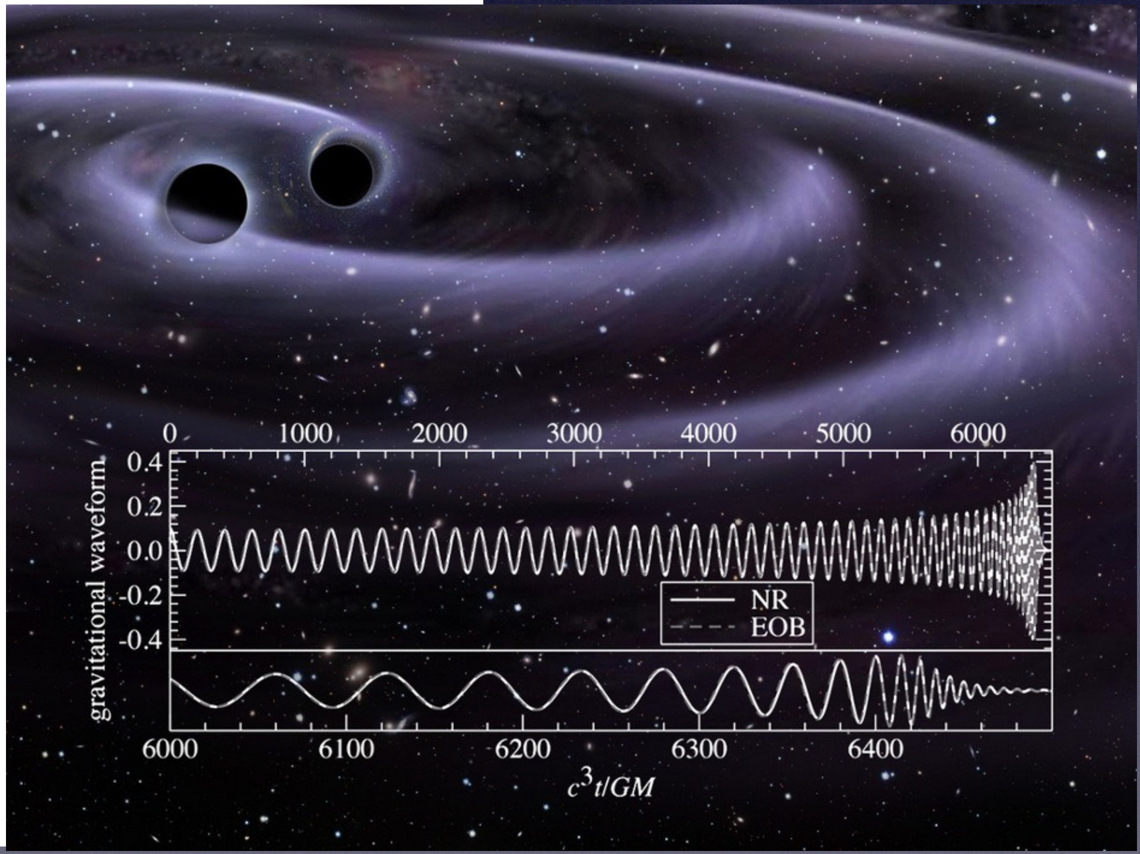
$$= (1 - h_{11})(2x_0)^2 = [1 - h_+ \cos(\omega t)]\,(2x_0)^2$$

$$dl \approx \left[1 - \frac{1}{2} h_+ \cos(\omega t)\right] (2x_0)$$

GW emission from a binary system (I)

- The binary consists of the two bodies M1 and M2 at distances a_1 and a_2 from the center of mass. The orbits are circular and lie on the x-y plane. The orbital angular frequency is Ω.





- We also define: $a = a_1 + a_2, \qquad \mu = M_1 M_2 / M,$

GW emission from a binary system (II)

- The only non-vanishing components of the quadrupole tensor are :

$$Q_{xx} = -Q_{yy} = (a_1^2 M_1 + a_2^2 M_2)\cos^2(\Omega t) = \frac{1}{2}\mu a^2 \cos(2\Omega t)$$

$$Q_{xy} = Q_{yx} = \frac{1}{2}\mu a^2 \sin(2\Omega t)$$

(GW frequency = 2Ω)

- And the GW luminosity of the system is (we use Kepler's 3rd law $\Omega^2 = GM/a^3$)

$$L_{\mathrm{GW}} = -\frac{dE}{dt} = \frac{G}{5c^5}(\mu\Omega a^2)^2 \langle\, 2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)\,\rangle$$

$$= \frac{32}{5}\frac{G}{c^5}\mu^2 a^4 \Omega^6 = \frac{32}{5}\frac{G^4}{c^5}\frac{M^3\mu^2}{a^5}$$

GW emission from a binary system (III)

- The total energy of the binary system can be written as :

$$E = \frac{1}{2}\Omega^2\left(M_1 a_1^2 + M_2 a_2^2\right) - \frac{G M_1 M_2}{a} = -\frac{1}{2}\frac{G\mu M}{a}$$

- As the gravitating system loses energy by emitting radiation, the distance between the two bodies shrinks at a rate:

$$\frac{dE}{dt} = \frac{G\mu M}{2a^2}\frac{da}{dt} \quad \rightarrow \quad \frac{da}{dt} = -\frac{64}{5}\frac{G^3}{c^5}\frac{\mu M}{a^3}$$

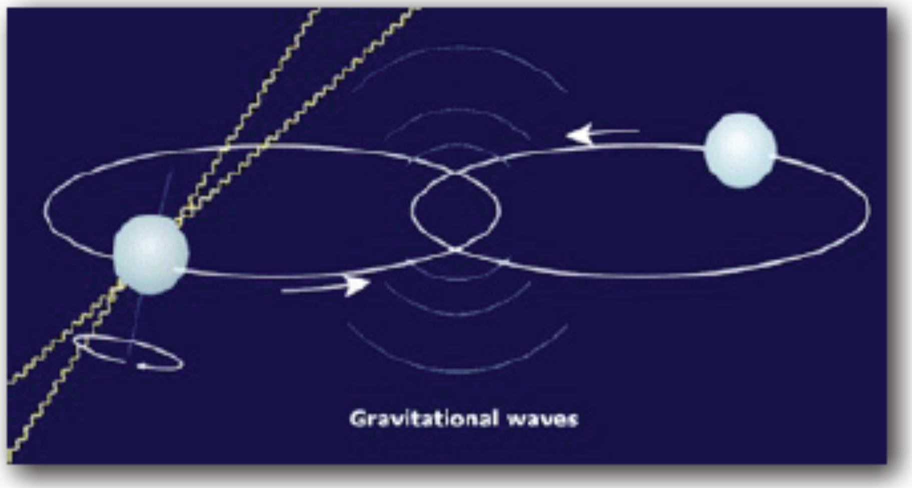
- The orbital frequency increases accordingly $\dot{T}/T = (3/2)\dot{a}/a$.

(initial separation)

- The system will coalesce after a time: $\tau = \frac{5}{256}\frac{c^5}{G^3}\frac{a_0^4}{\mu M^4}$

PSR 1913+16: a Nobel-prize GW source

- The now famous Hulse & Taylor binary neutron star system provided the first astrophysical evidence of the existence of GWs !



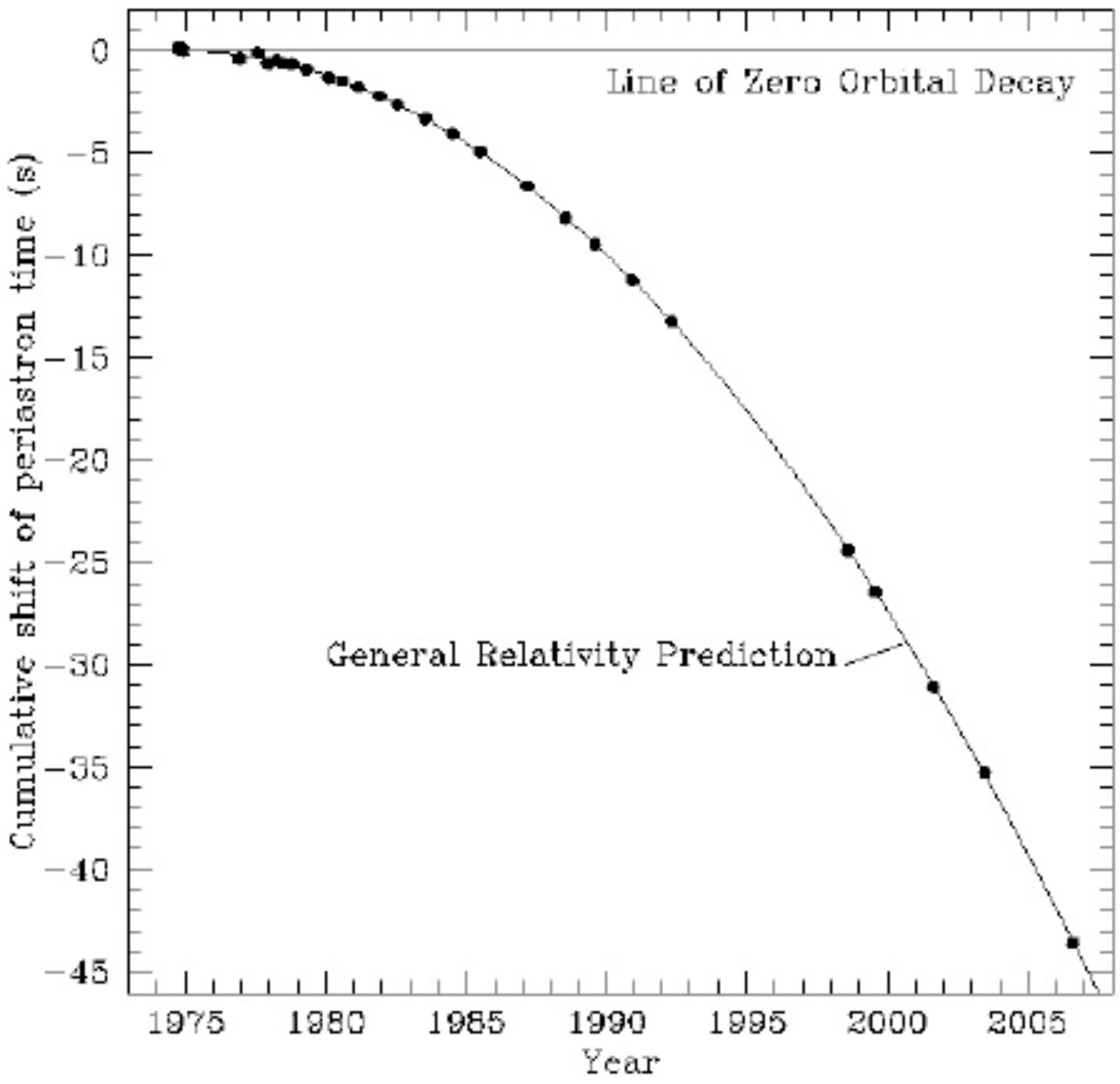
- The system's parameters: $r = 5\,\mathrm{Kpc}, \quad M_1 \approx M_2 \approx 1.4\,M_\odot, \quad T = 7\,\mathrm{h}\;\; 45\,\mathrm{min}$

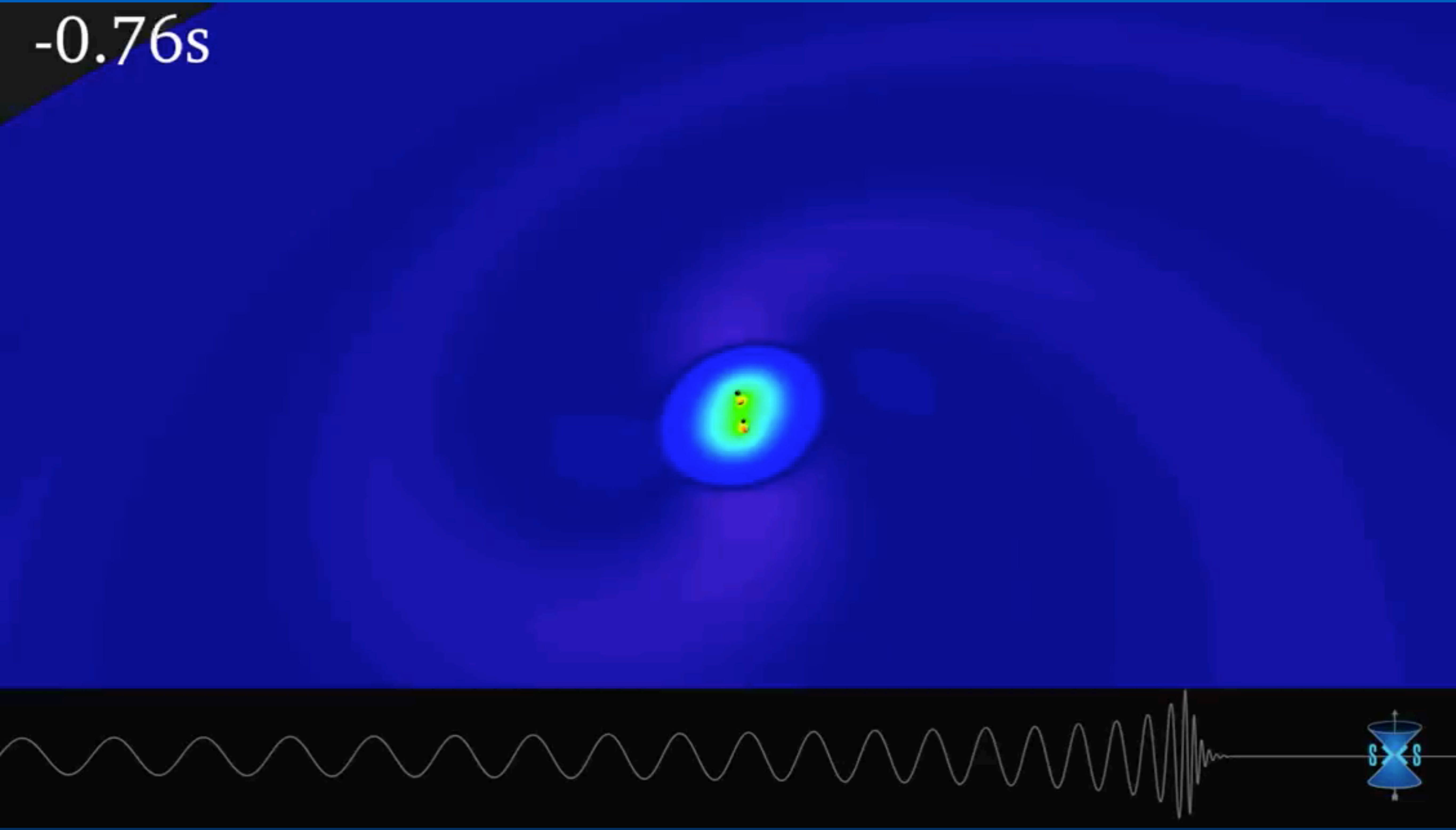
- Using the previous equations we can predict:

$$\dot{T} = -2.4 \times 10^{-12}\,\mathrm{sec/sec}, \qquad f_{\mathrm{GW}} = 7 \times 10^{-5}\,\mathrm{Hz}, \qquad h \sim 10^{-23}, \quad \tau \approx 3.5 \times 10^{8}\,\mathrm{yr}$$

Theory vs observations

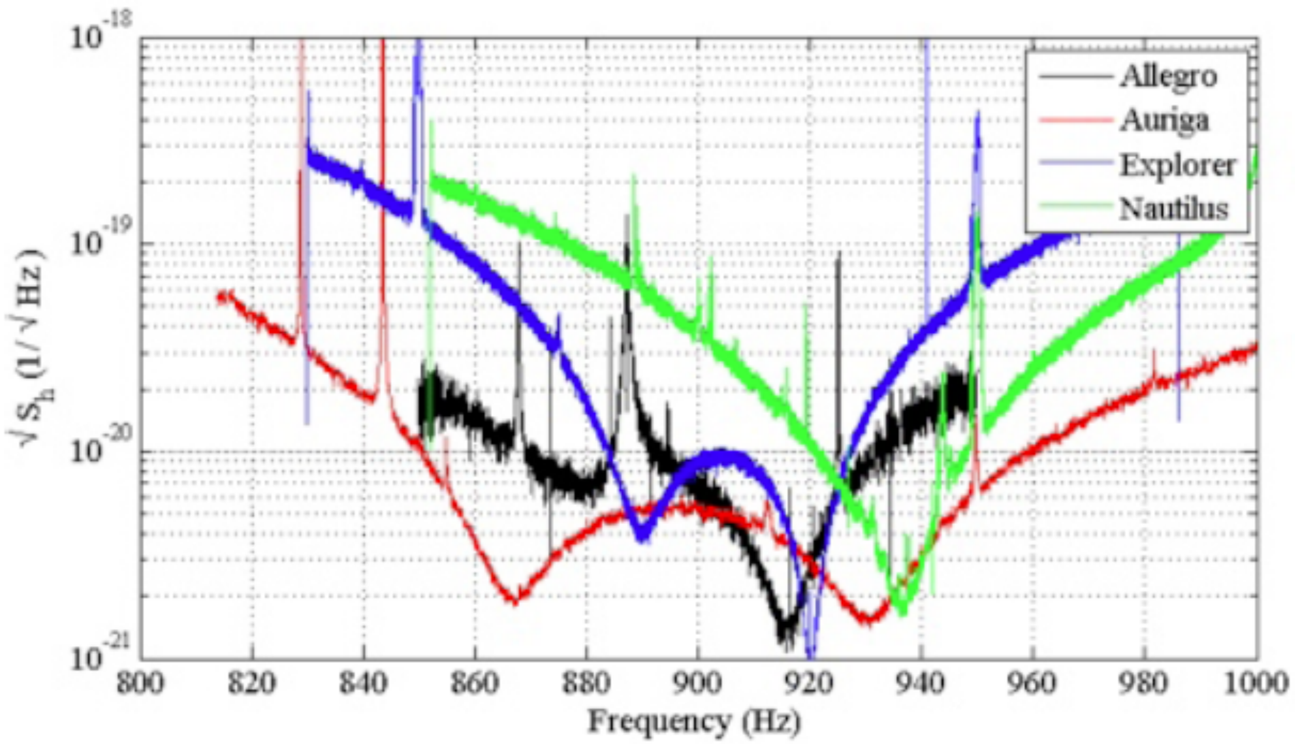
- How can the orbital parameters be measured with such high precision?
- One of the neutron stars is a pulsar, emitting extremely stable periodic radio pulses. The emission is modulated by the orbital motion.
- Since the discovery of the H-T system in 1974 more such binaries were found by astronomers.




-0.76s

Bar detectors

- Bar detectors are narrow bandwidth instruments (like the previous toy-model)

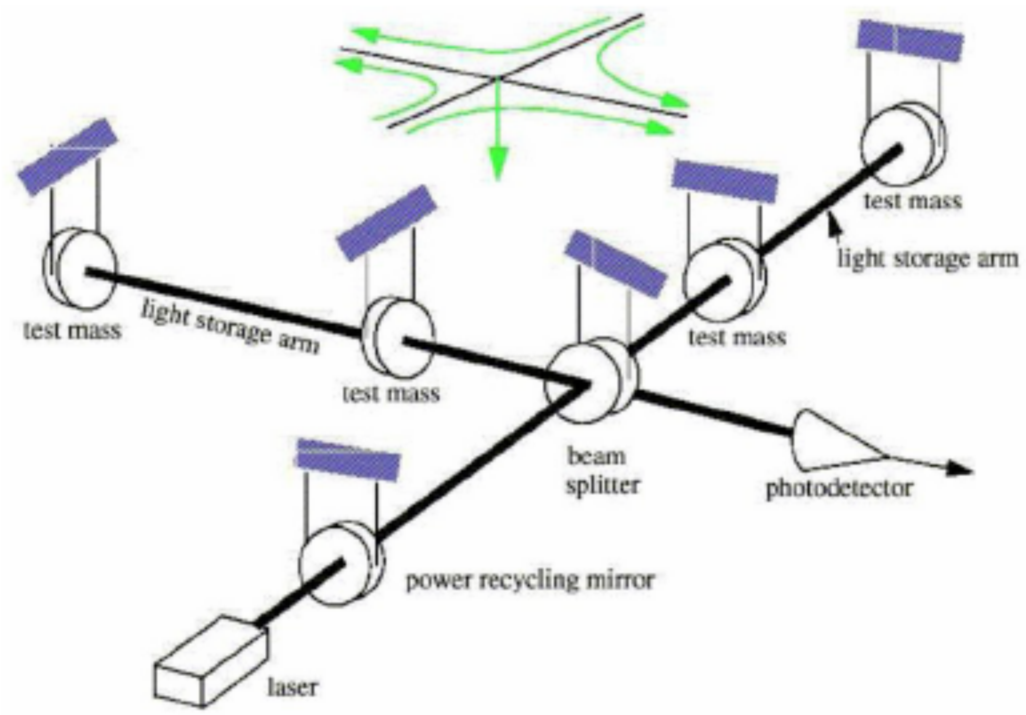


Sensitivity curves of various bar detectors

Joseph Weber

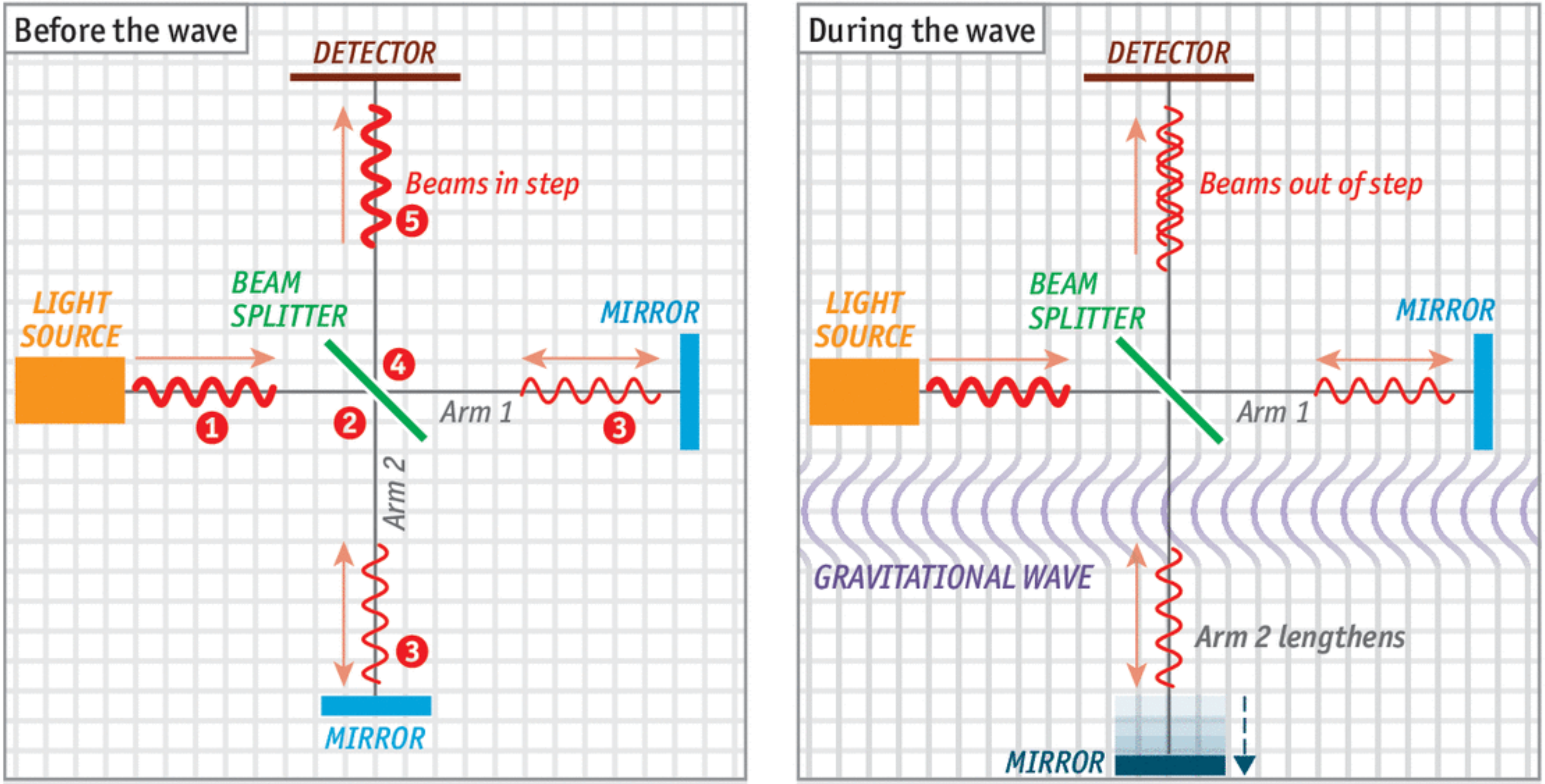
Detectors: laser interferometry

- A laser interferometer is an alternative choice for GW detection, offering a combination of very high sensitivities over a broad frequency band.
- Suspended mirrors play the role of "test-particles", placed in perpendicular directions. The light is reflected on the mirrors and returns back to the beam splitter and then to a photodetector where the fringe pattern is monitored.



Catching a wave

How a laser-interferometer observatory works

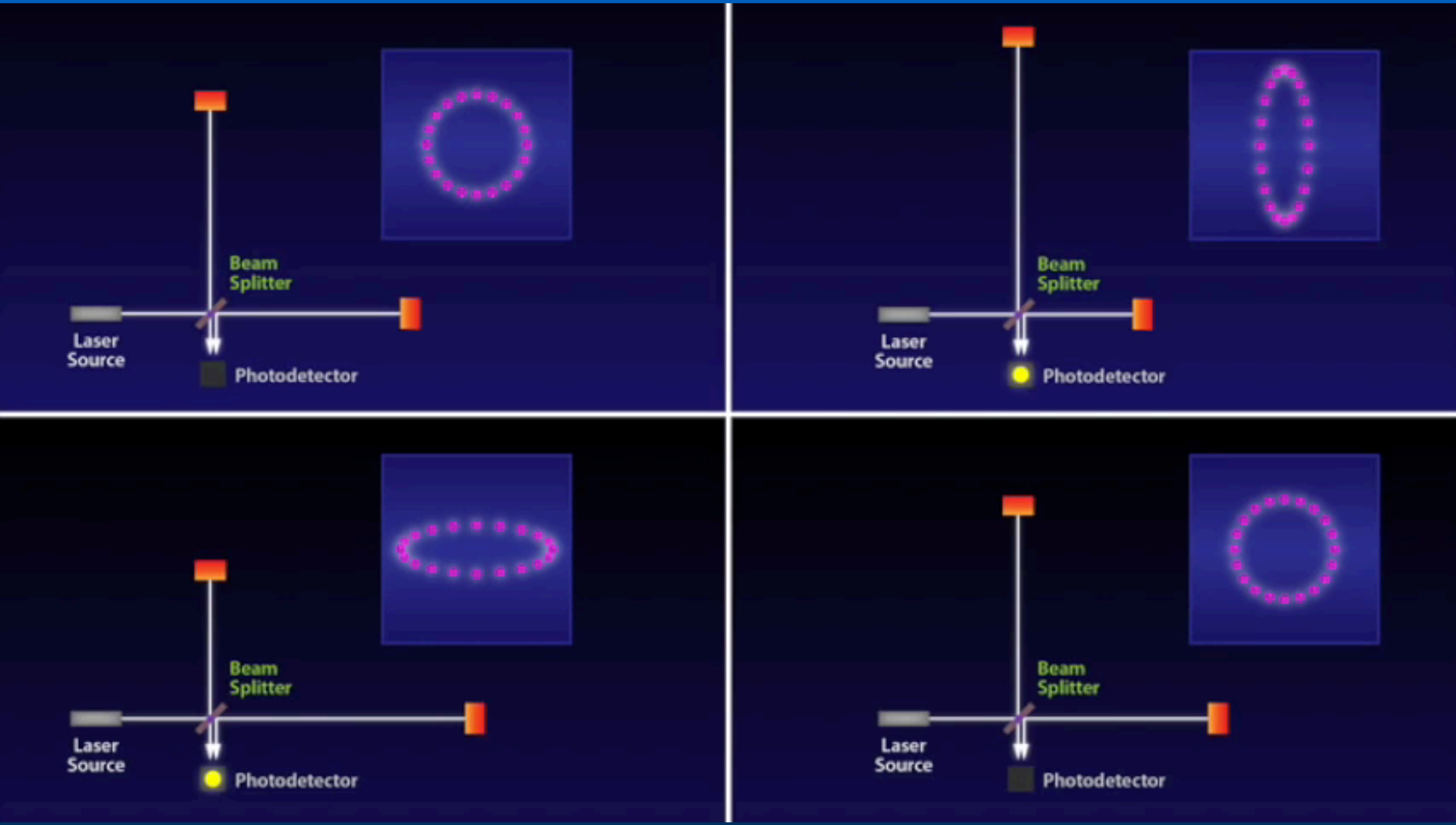


The light source sends out a beam 1 that is divided by a beam splitter 2. The half-beams produced follow paths of identical length 3, reflecting off mirrors to recombine 4, then travel in step to the detector 5.

When a *gravitational wave* arrives, it disturbs space-time, lengthening (in this example) the light's path along **arm 2**; when the beams recombine and arrive at the detector, they are no longer in step.

Source: *The Economist*

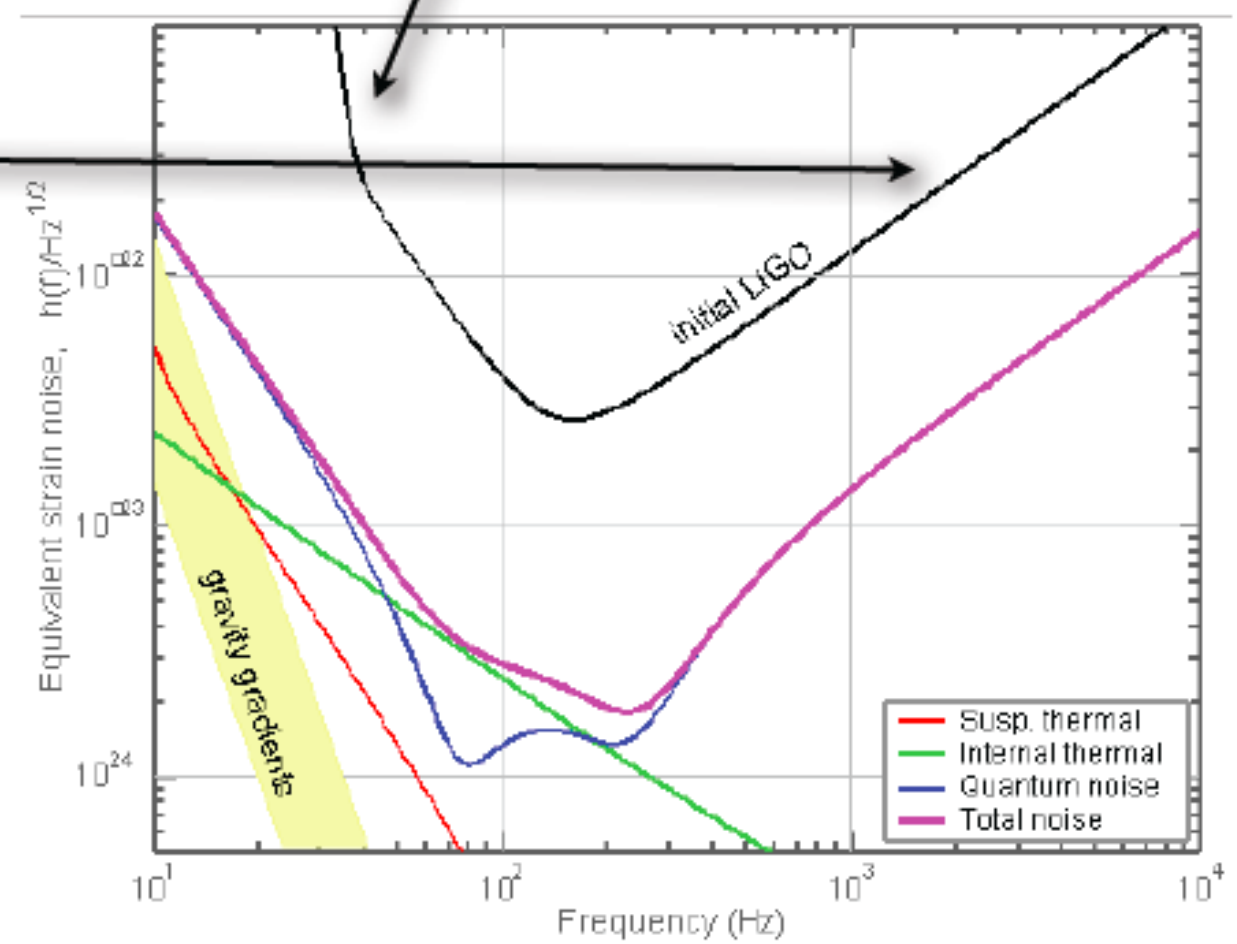
Economist.com


Beam
Splitter
Laser
Source
Photodetector
Beam
Splitter
Laser
Source
Photodetector
Beam
Splitter
Laser
Source
Photodetector
Beam
Splitter
Laser
Source
Photodetector

Noise in interferometric detectors

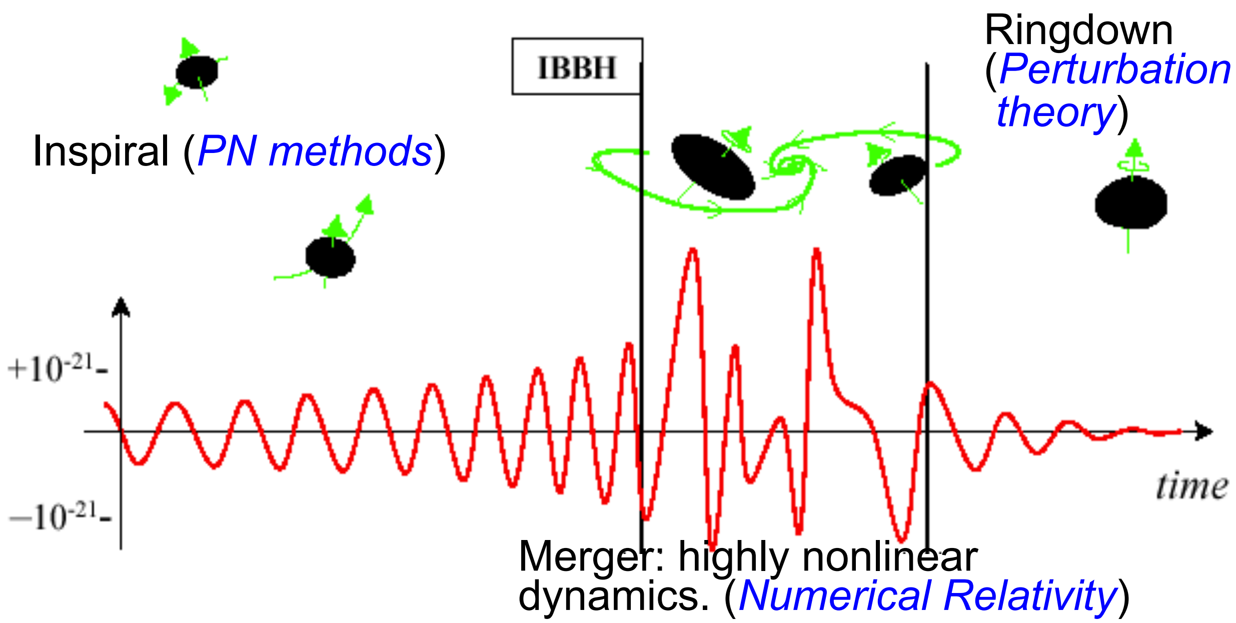
- Seismic noise (low frequencies). At frequencies below 60 Hz, the noise in the interferometers is dominated by seismic noise. The vibrations of the ground couple to the mirrors via the wire suspensions which support them. This effect is strongly suppressed by properly designed suspension systems. Still, seismic noise is very difficult to eliminate at frequencies below 5-10 Hz.

- Photon shot noise (high frequencies). The precision of the measurements is restricted by fluctuations in the fringe pattern due to fluctuations in the number of detected photons. The number of detected photons is proportional to the intensity of the laser beam. Statistical fluctuations in the number of detected photons imply an uncertainty in the measurement of the arm length.

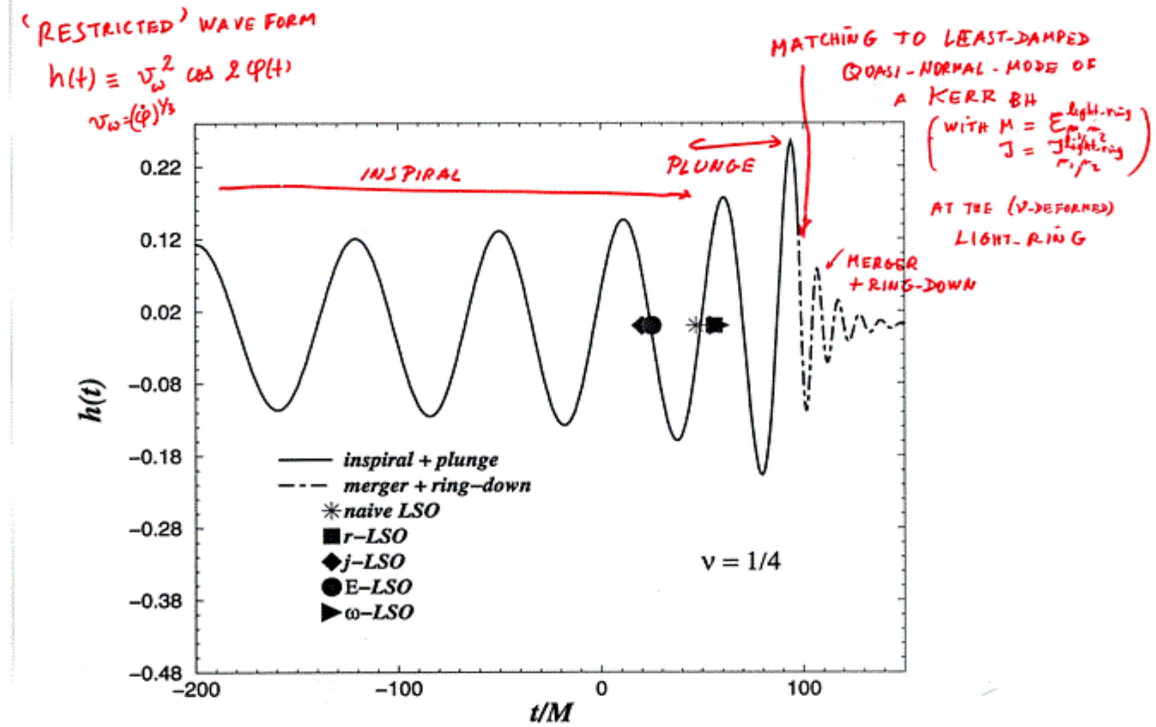


Templates for GWs from BBH coalescence

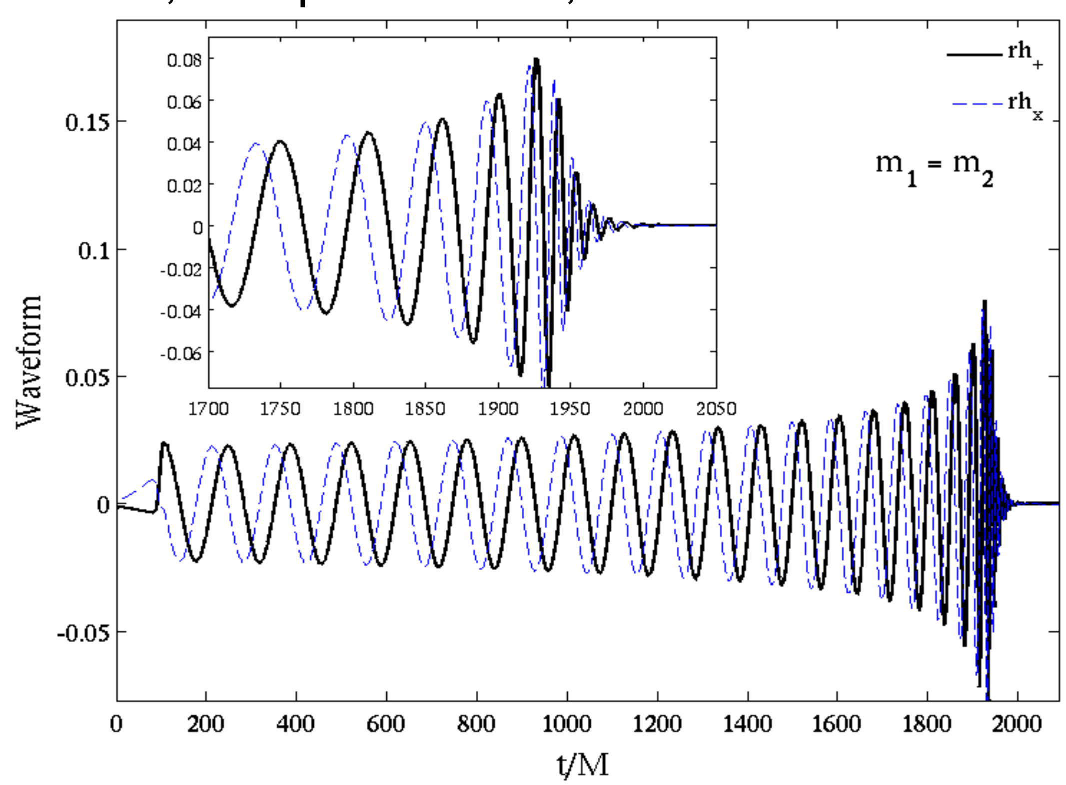
(Brady, Craighton, Thorne 1998)



(Buonanno & Damour 2000)



Numerical Relativity, the 2005 breakthrough:
Pretorius, Campanelli et al., Baker et al. …



A catalog of 171 high-quality binary black-hole simulations for gravitational-wave astronomy

[arXiv: 1304.6077]

Abdul H. Mroué,[1] Mark A. Scheel,[2] Béla Szilágyi,[2] Harald P. Pfeiffer,[1] Michael Boyle,[3] Daniel A. Hemberger,[3] Lawrence E. Kidder,[3] Geoffrey Lovelace,[4,2] Sergei Ossokine,[1,5] Nicholas W. Taylor,[2] Anıl Zenginoğlu,[2] Luisa T. Buchman,[2] Tony Chu,[1] Evan Foley,[4] Matthew Giesler,[4] Robert Owen,[6] and Saul A. Teukolsky[3]

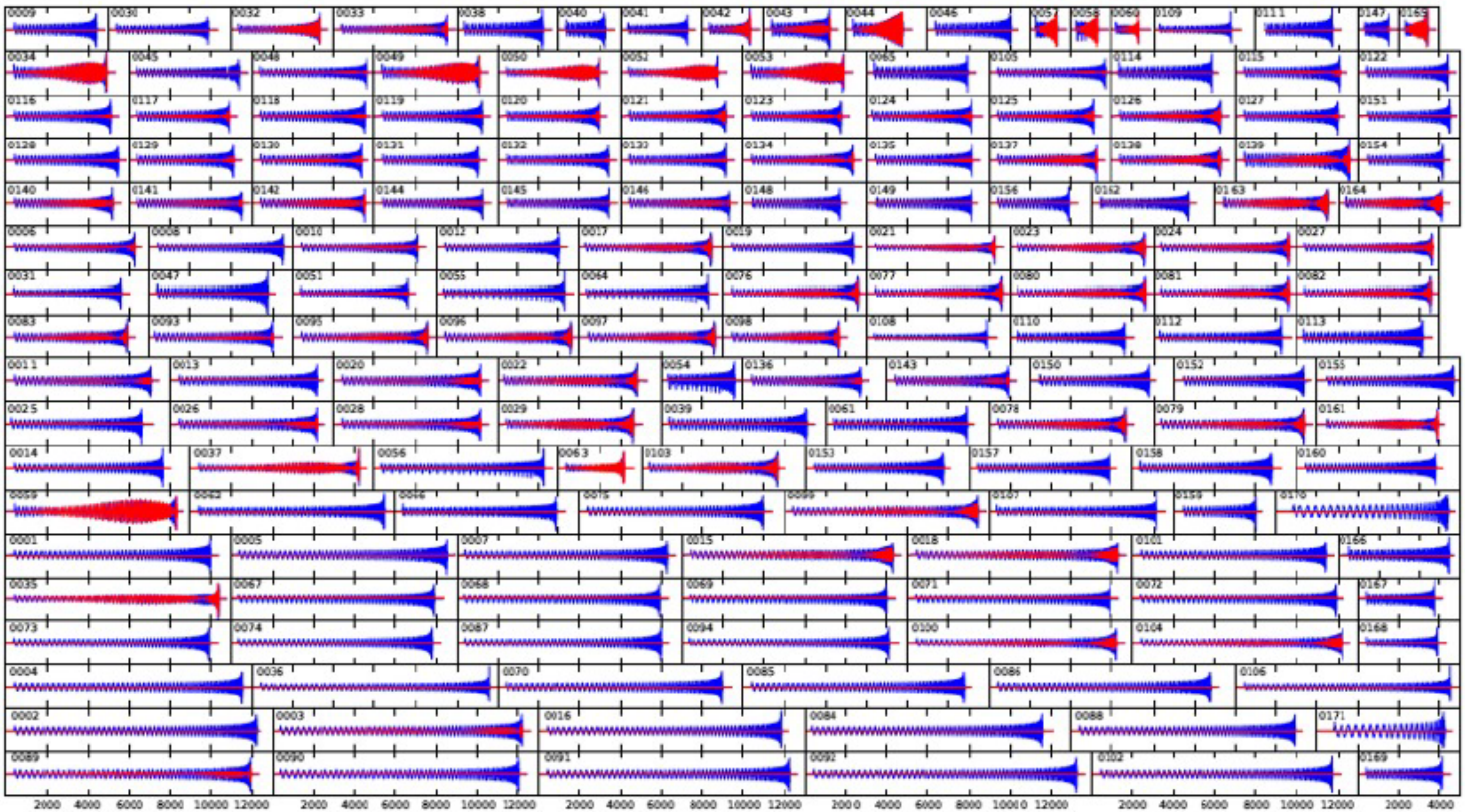
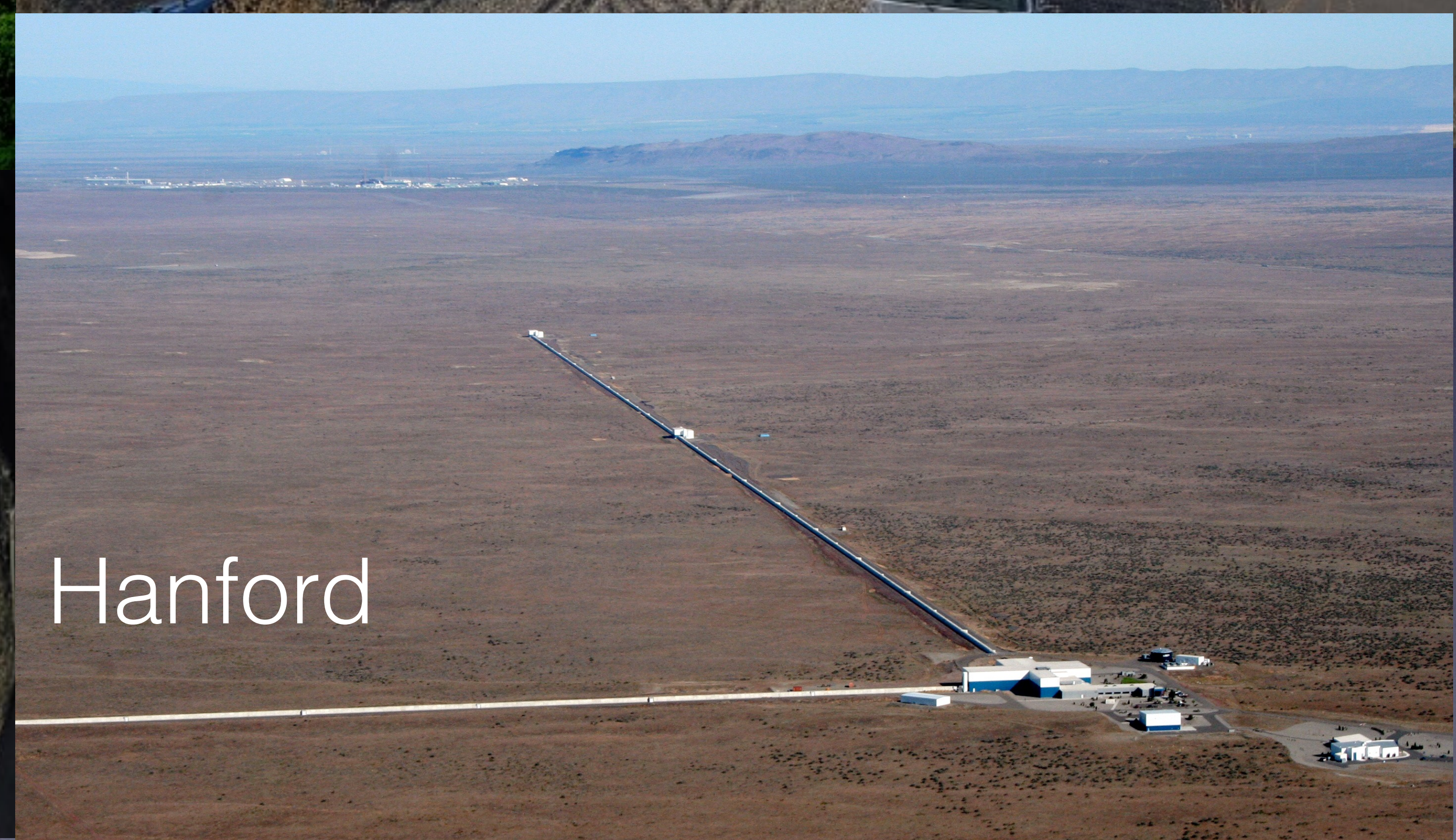


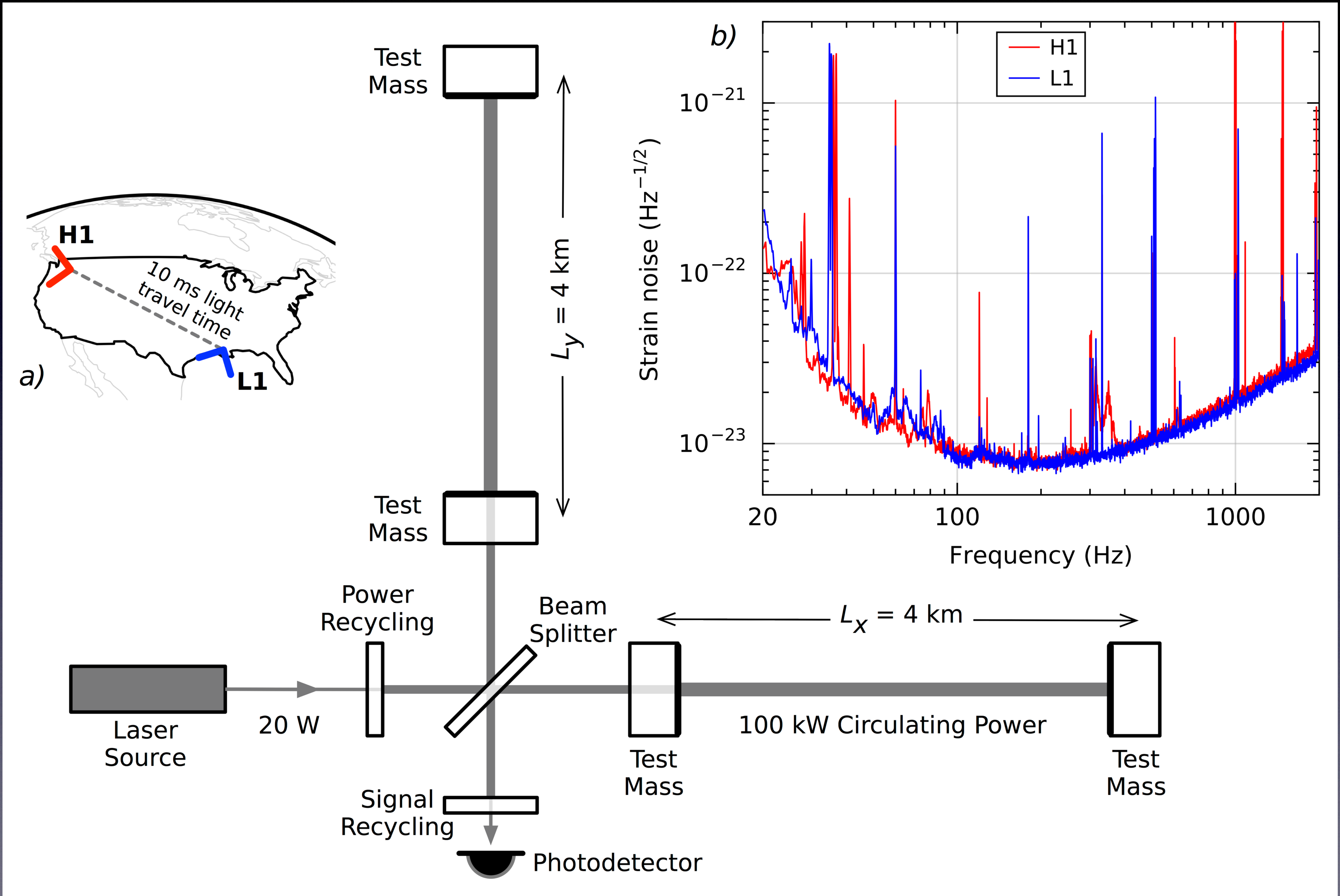
FIG. 3: Waveforms from all simulations in the catalog. Shown here are h_+ (blue) and $h_\times$ (red) in a sky direction parallel to the initial orbital plane of each simulation. All plots have the same horizontal scale, with each tick representing a time interval of $2000M$, where M is the total mass.

Detectors: the present (I)

The twin LIGO detectors (L = 4 km) at Livingston Louisiana and Hanford Washington (US).

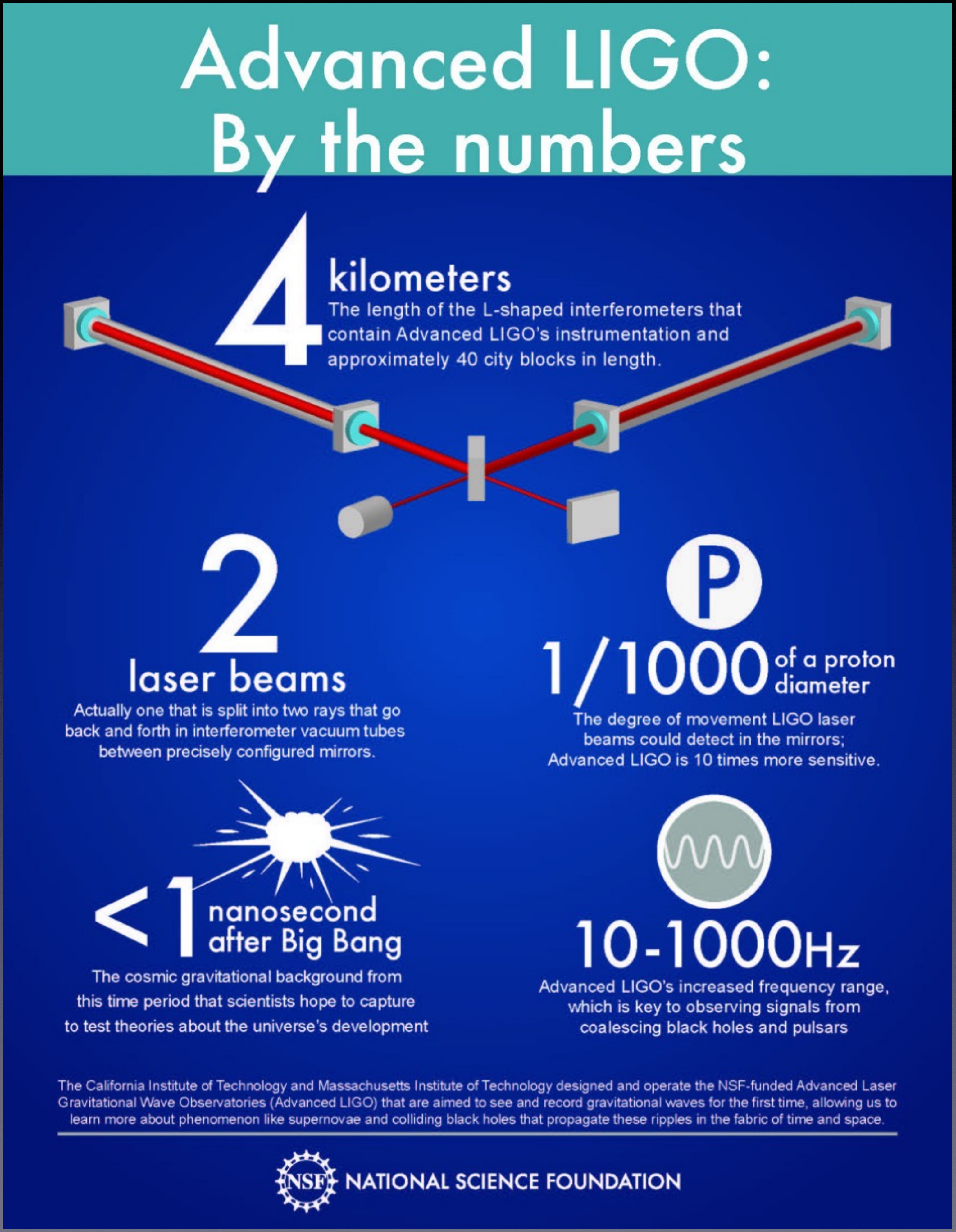

Livingston



Hanford

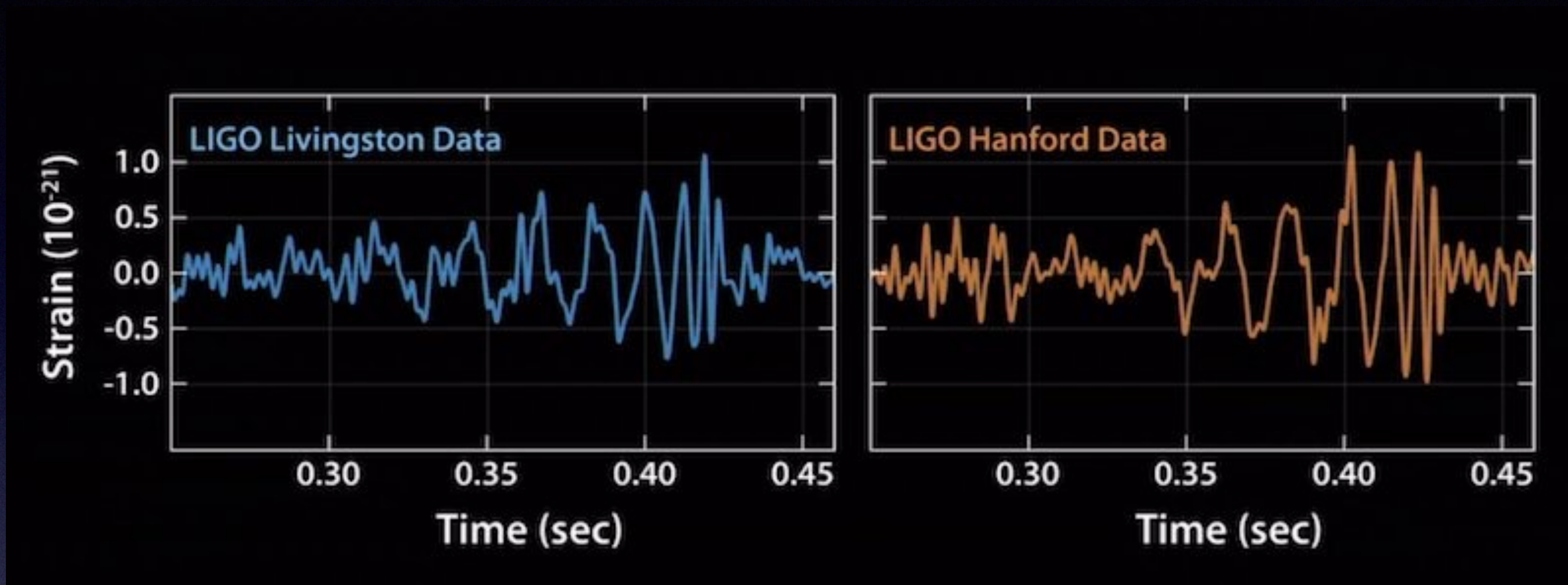


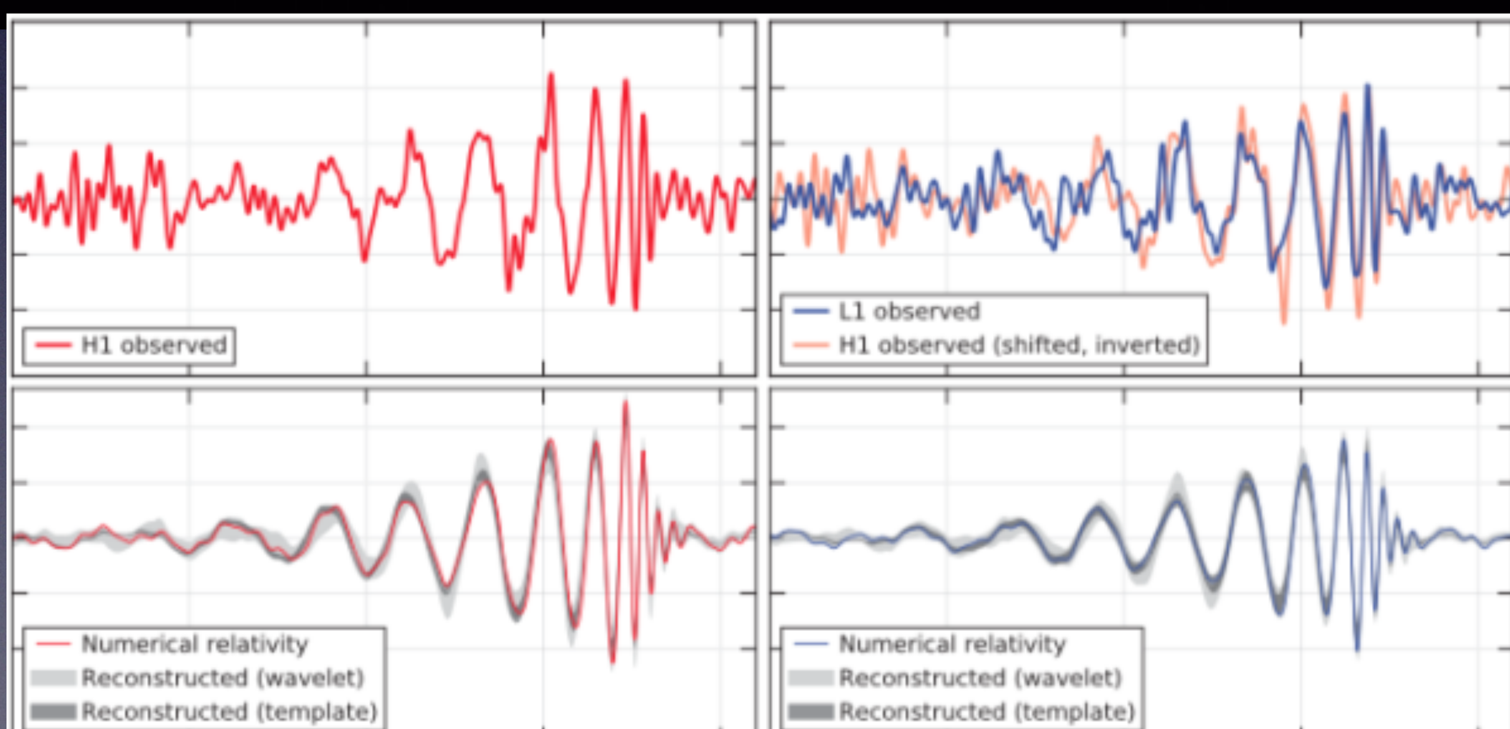
a)
H1
L1
10 ms light travel time
b)
H1
L1
Strain noise (Hz$^{-1/2}$)
10^{-21}
10^{-22}
10^{-23}
20
100
1000
Frequency (Hz)
Test Mass
Test Mass
L_y = 4 km
Power Recycling
Beam Splitter
L_x = 4 km
Laser Source
20 W
100 kW Circulating Power
Test Mass
Test Mass
Signal Recycling
Photodetector

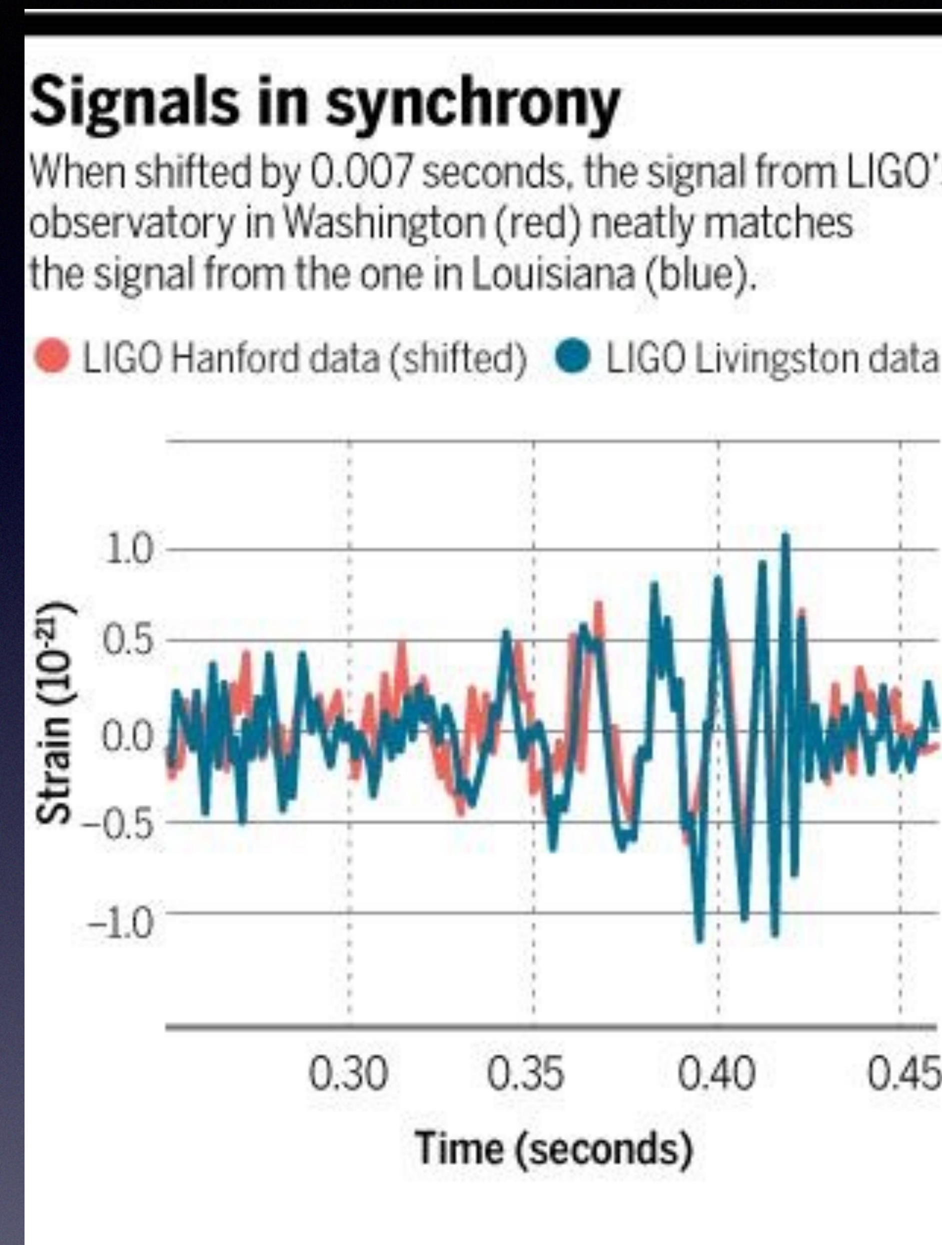
LIGO's interferometer is classified as a Dual Recycled, Fabry-Perot Michelson Interferometer.



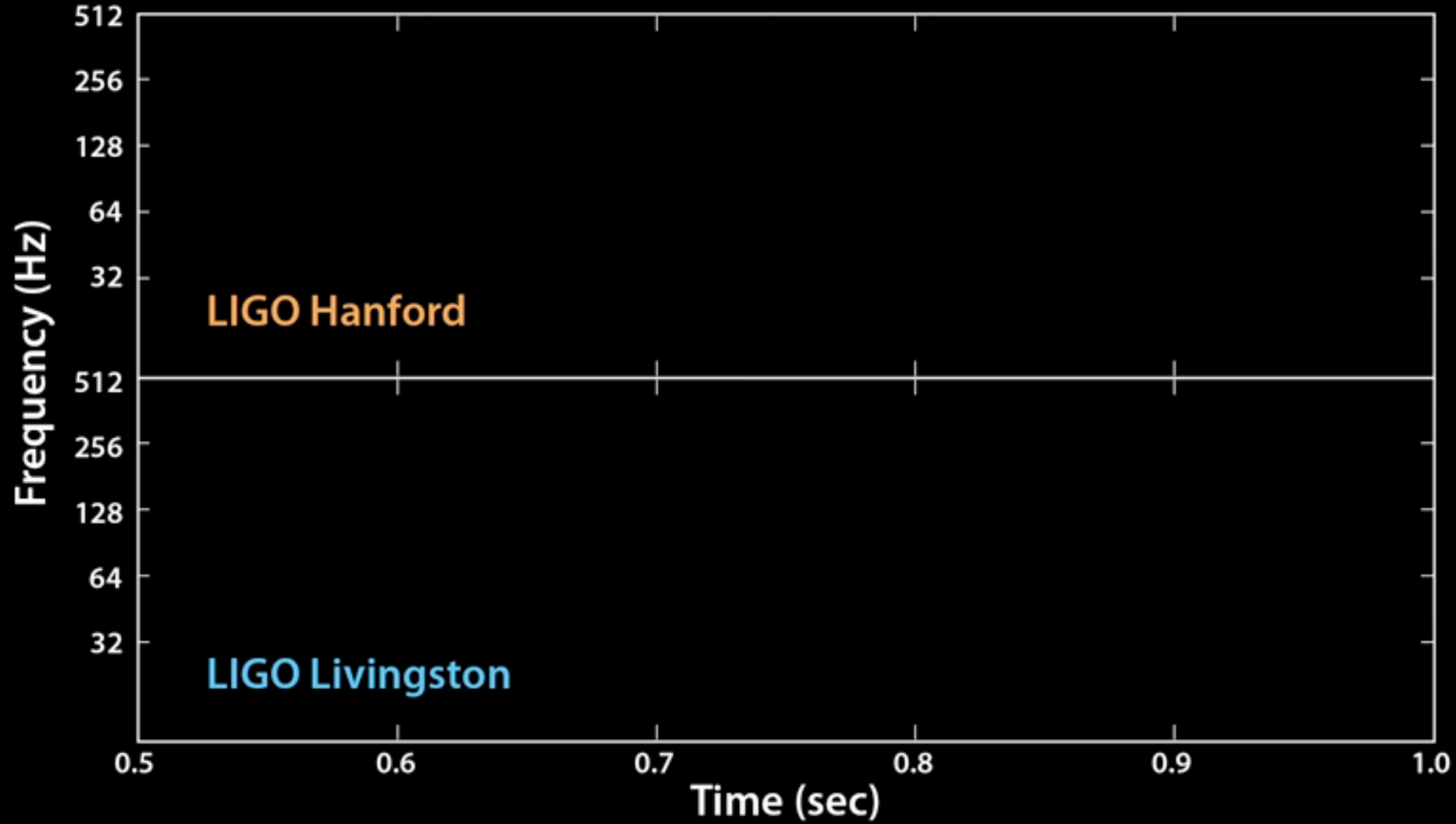
Gravitational waves detected by LIGO!



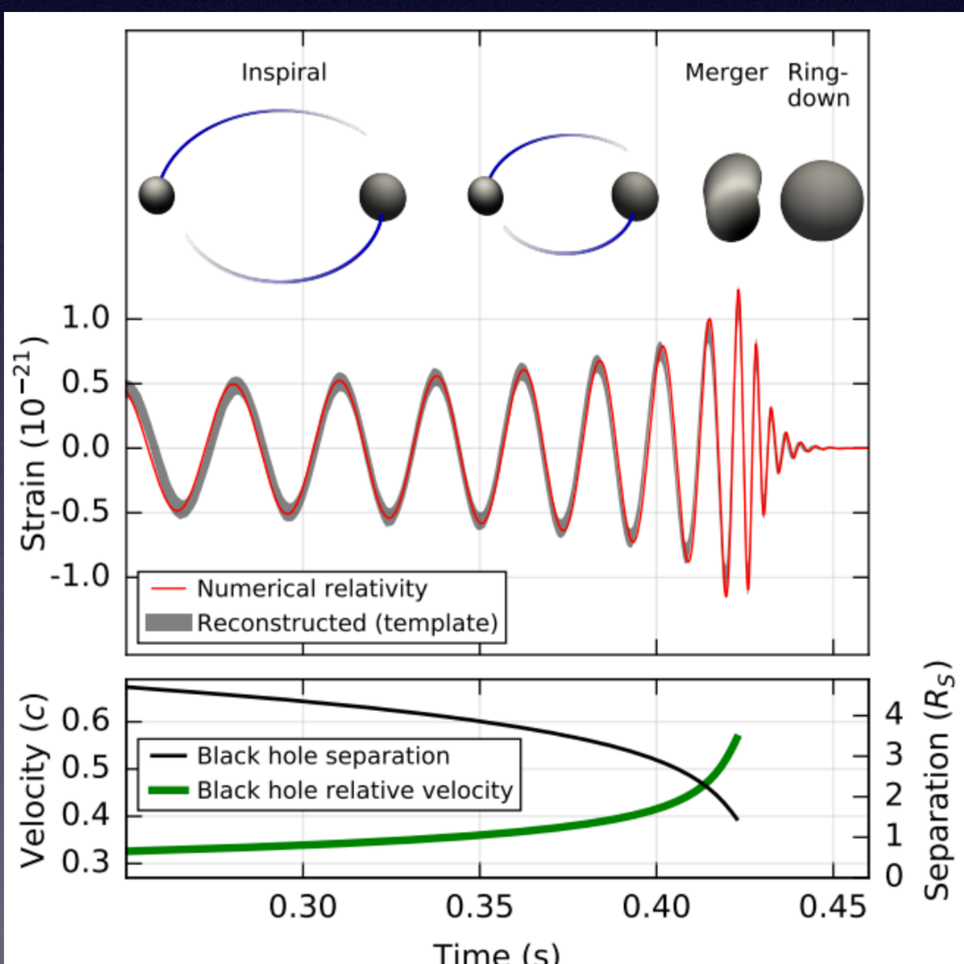


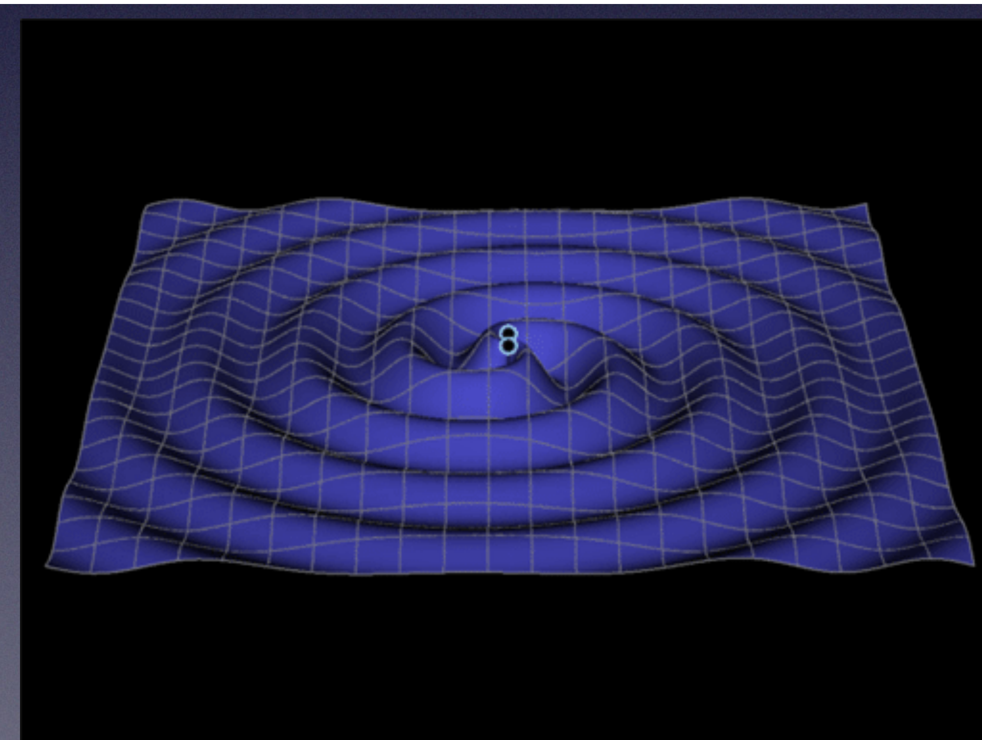


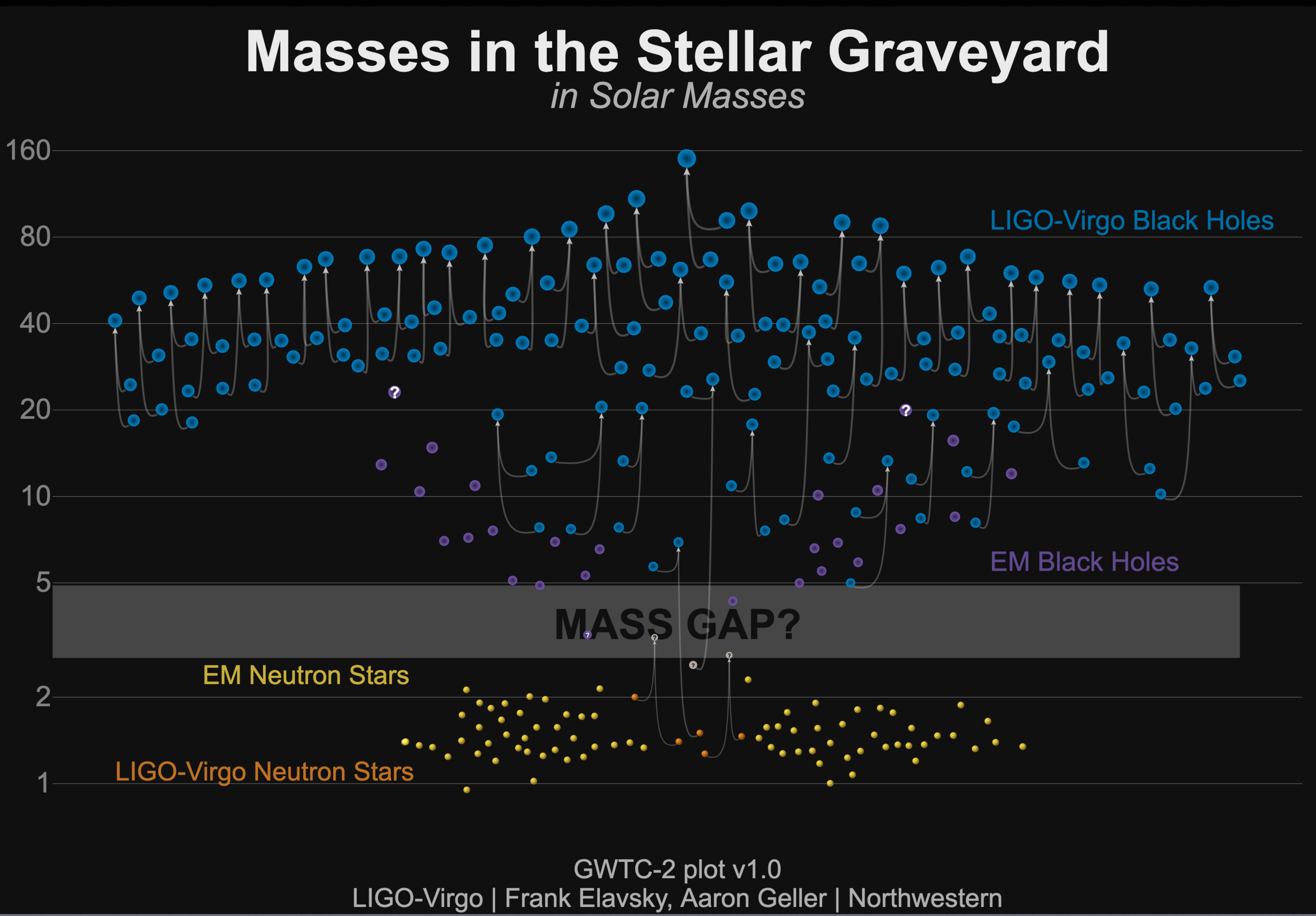
September 14th, 2015, 09:50:45 UTC.
Range: from 35 to 250 Hz


512
256
128
64
32
LIGO Hanford
512
256
128
64
32
LIGO Livingston
Frequency (Hz)
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)

Primary black hole mass	$36^{+5}_{-4} M_\odot$
Secondary black hole mass	$29^{+4}_{-4} M_\odot$
Final black hole mass	$62^{+4}_{-4} M_\odot$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$



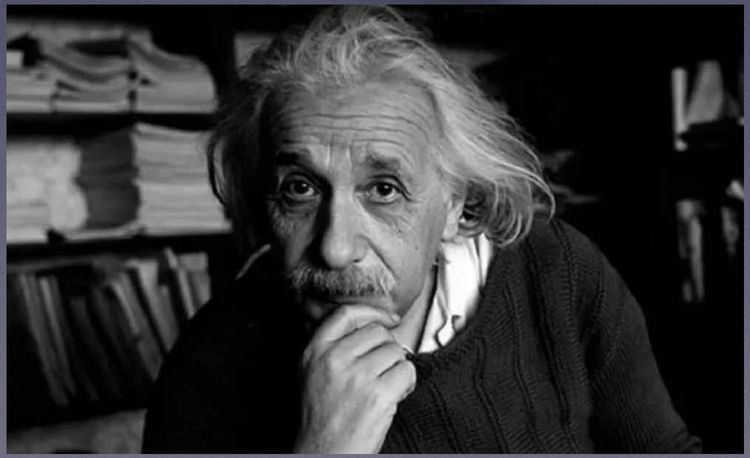



Masses in the Stellar Graveyard
in Solar Masses
160
80
40
20
10
5
2
1
LIGO-Virgo Black Holes
EM Black Holes
MASS GAP?
EM Neutron Stars
LIGO-Virgo Neutron Stars
GWTC-2 plot v1.0
LIGO-Virgo | Frank Elavsky, Aaron Geller | Northwestern

Implications of the detection of GWs:

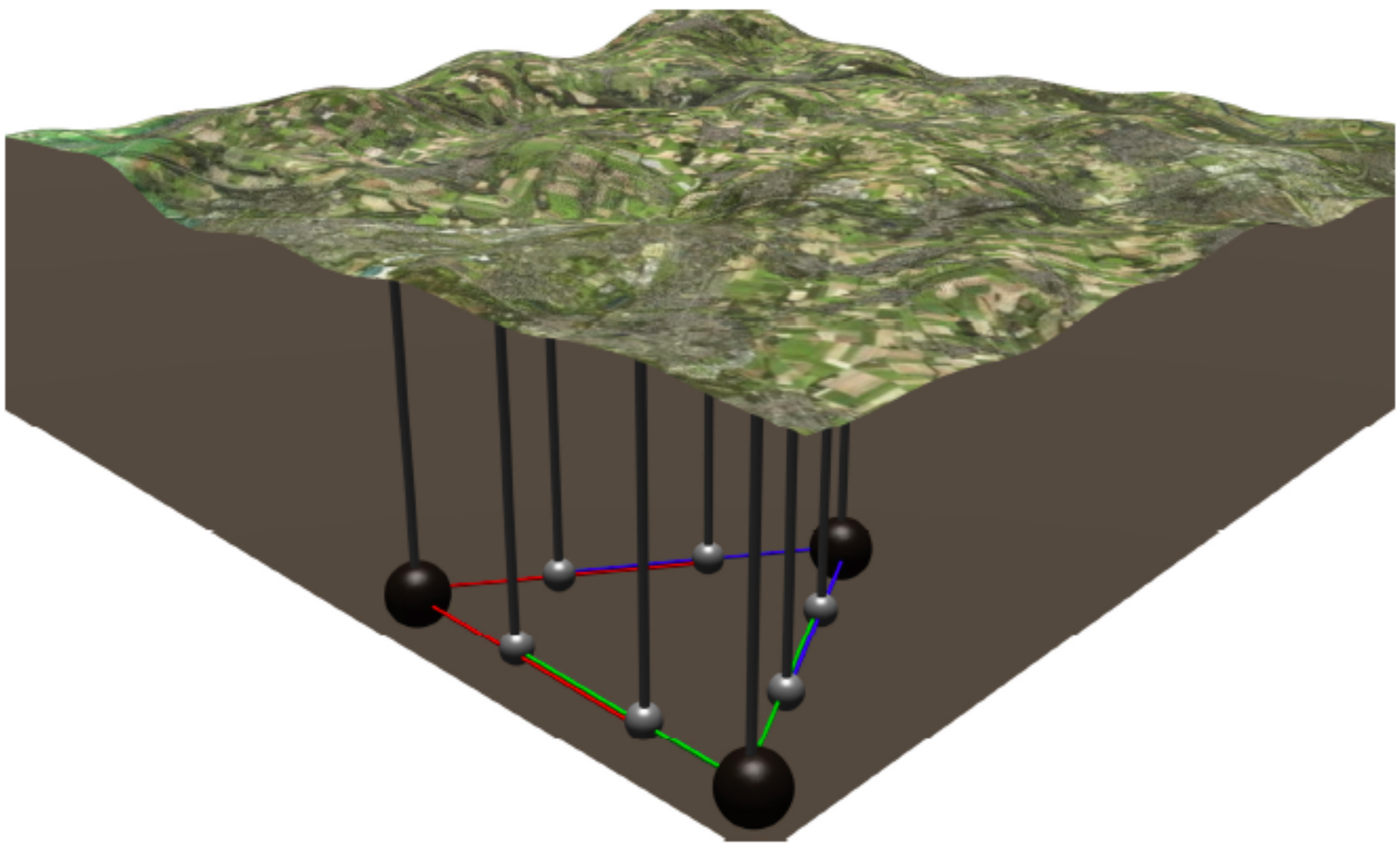
- Gravitational waves exist
- Compact objects very much like to black holes exist
- Gravitational waves transport energy —> the spacetime has energy in absence of matter and can be considered as material as well, since it can act upon physical systems such as mirrors.
- Spacetime has a dimensionality of n=4 or higher.
- Existence is non-local.

Gravitational wave astronomy is born!



Going underground: the ET

- The Einstein Telescope will be the next generation underground detector.



The Einstein Telescope has been proposed by 8 European research institutes:

European Gravitational Observatory
Istituto Nazionale di Fisica Nucleare
Max Planck Society
Centre National de la Recherche Scientifique
University of Birmingham
University of Glasgow
NIKHEF
Cardiff University

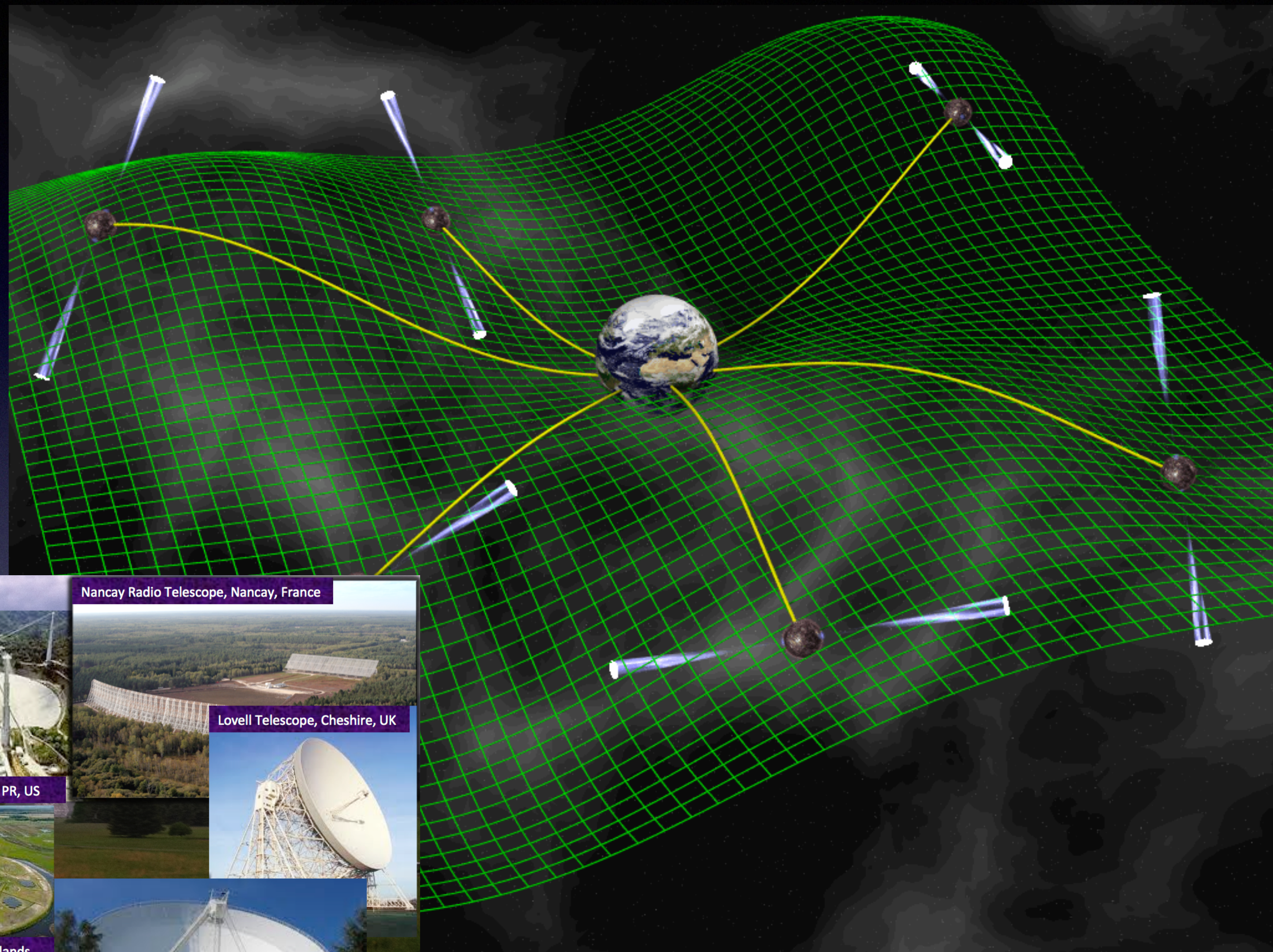
The arms will be 10 km long (compared to 4 km for LIGO, and 3 km for Virgo), and like LISA, there will be three arms in an equilateral triangle, with two detectors in each corner.

The low-frequency interferometers (1 to 250 Hz) will use optics cooled to 10 K (−441.7 °F; −263.1 °C), with a beam power of about 18 kW in each arm cavity. The high-frequency ones (10 Hz to 10 kHz) will use room-temperature optics and a much higher recirculating beam power of 3 MW.

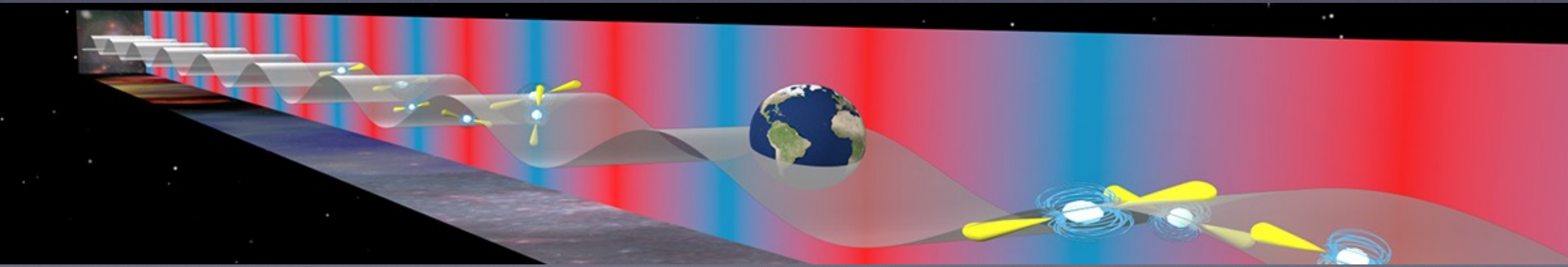
Gravitational wave detection with pulsars

EPTA/LEAP
IPTA

International Pulsar Timing Array

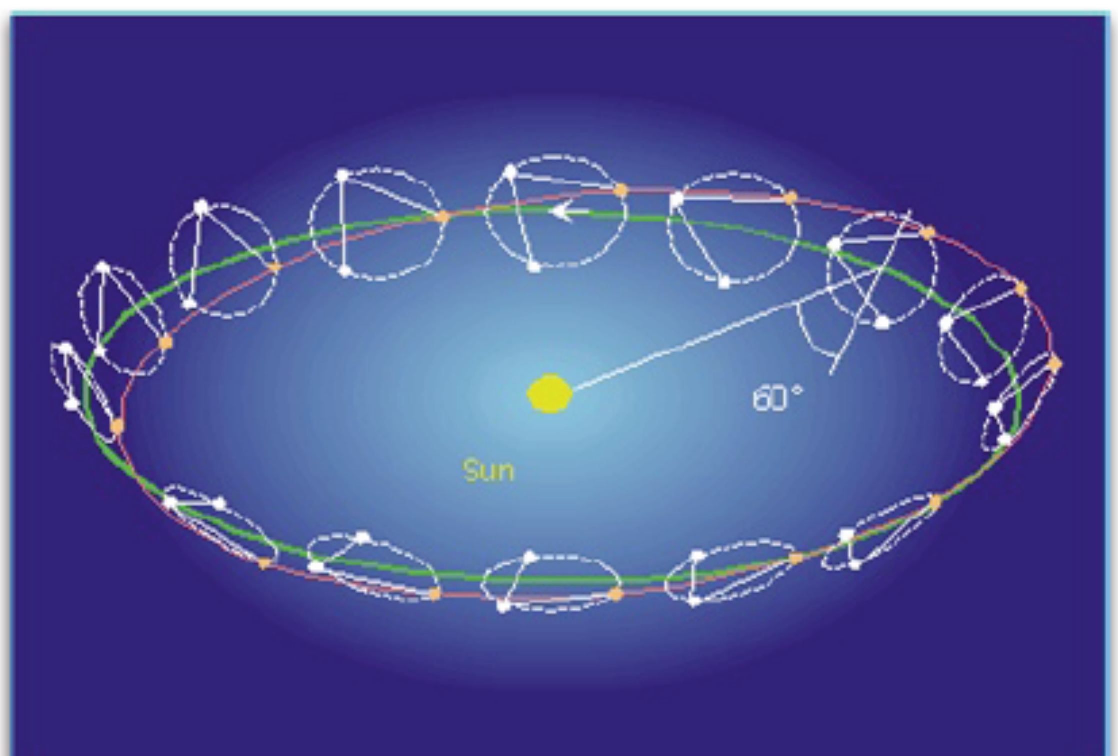


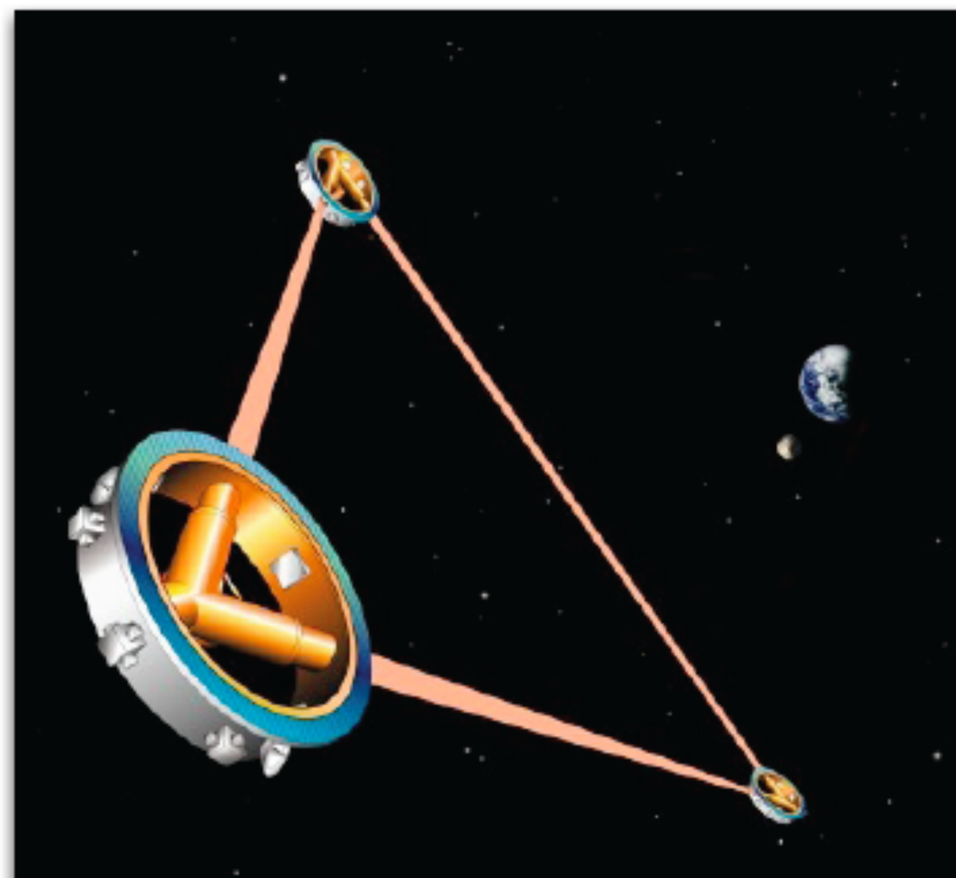
NANOGrav stands for North American Nanohertz Observatory for Gravitational Waves. As the name implies, NANOGrav members are drawn from across the United States and Canada . Their goal is to study the Universe using gravitational waves. NANOGrav uses the Galaxy itself to detect gravitational waves with the help of pulsars. This is known as a Pulsar Timing Array, or PTA. NANOGrav scientists make use of some of the world's best telescopes and most advanced technology, drawing on physics, computer science, signal processing, and electrical engineering.



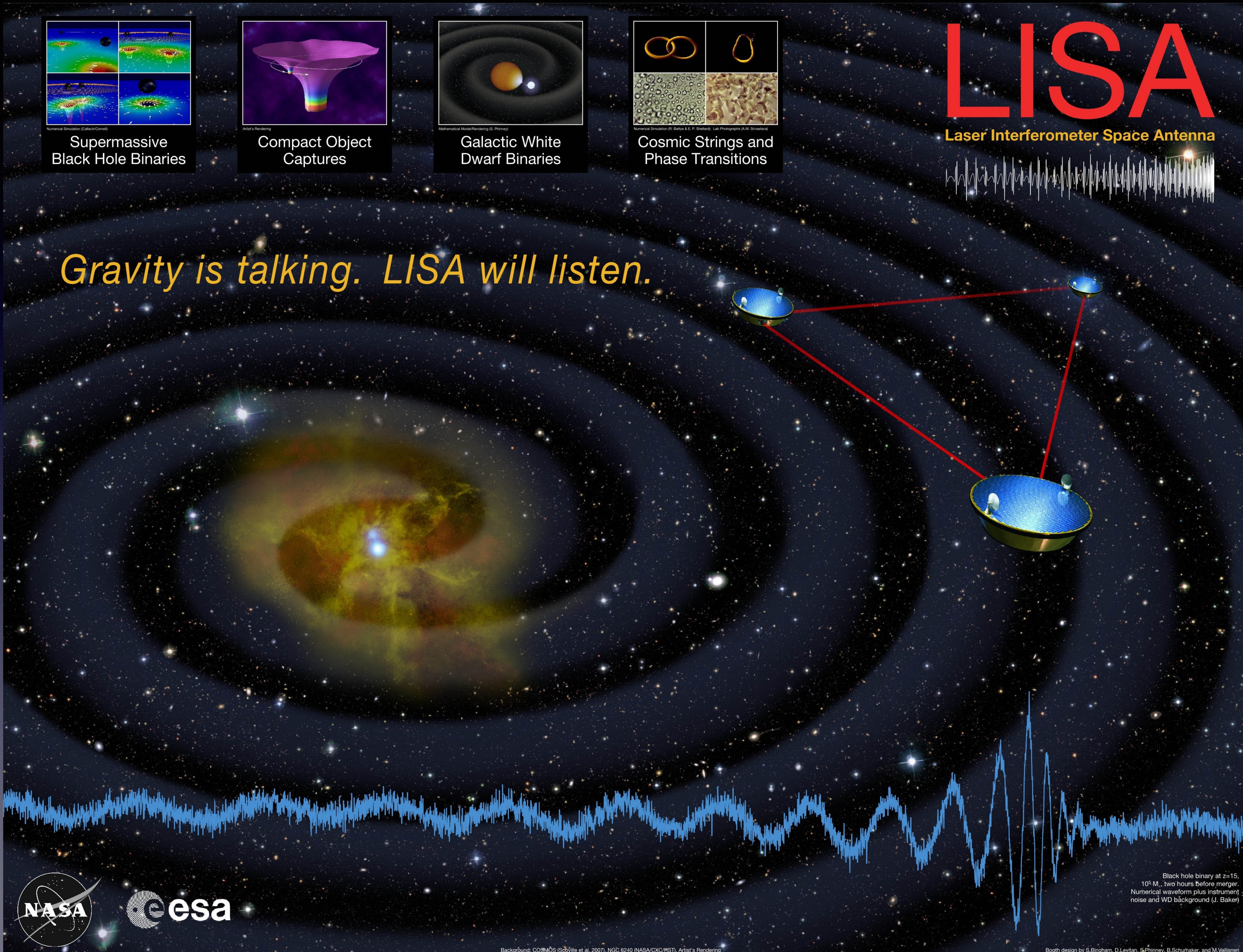
Going to space: the LISA detector

- Space-based detectors: “noise-free” environment, abundance of space!
- Long-arm baseline, low frequency sensitivity
- LISA: Up until recently a joint NASA/ESA mission, now an ESA mission only. To be launched around 2020.

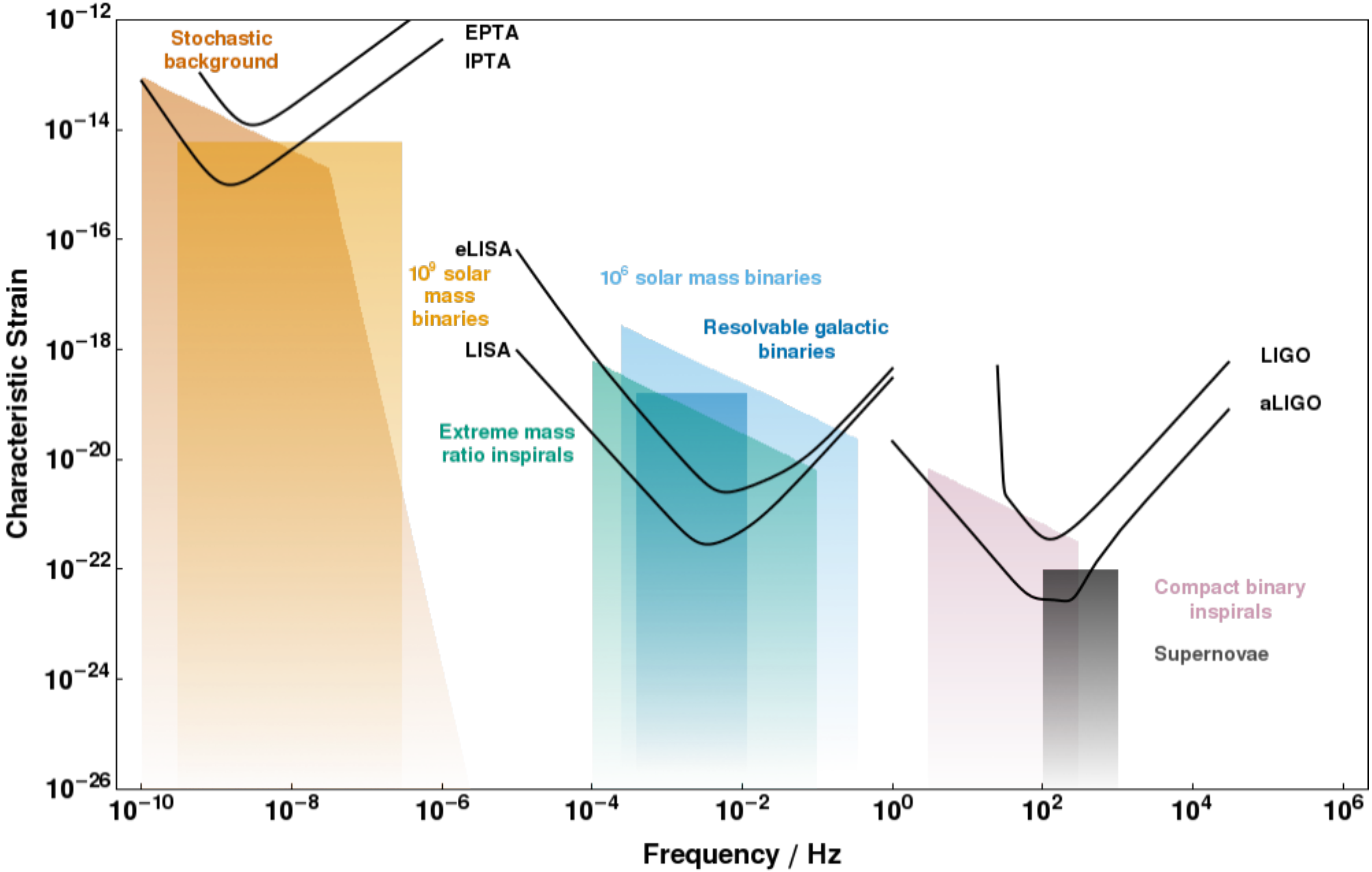




LISA (NASA-ESA)



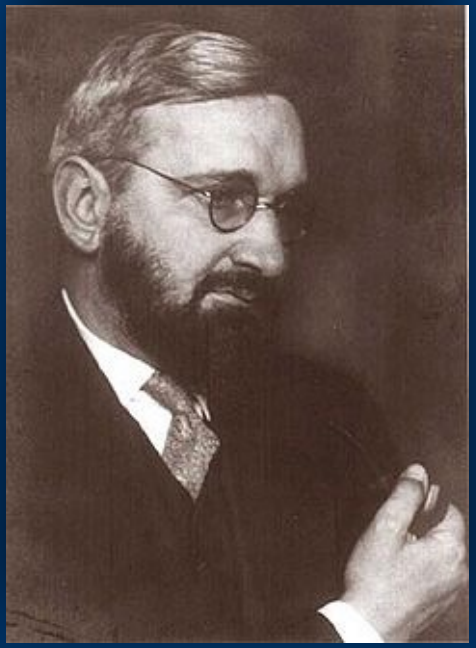
Numerical Simulation (Caltech/Cornell)
Supermassive Black Hole Binaries
Artist's Rendering
Compact Object Captures
Mathematical Model/Rendering (S. Phinney)
Galactic White Dwarf Binaries
Numerical Simulation (R. Battye & E. P. Shellard); Lab Photographs (A.M. Srivastava)
Cosmic Strings and Phase Transitions
LISA
Laser Interferometer Space Antenna
Gravity is talking. LISA will listen.
Black hole binary at z=15, 10^5 M☉, two hours before merger. Numerical waveform plus instrument noise and WD background (J. Baker)
NASA
esa
Background: COSMOS (Scoville et al. 2007), NGC 6240 (NASA/CXC/HST), Artist's Rendering
Booth design by S.Bingham, D.Levitan, S.Phinney, B.Schumaker, and M.Vallisneri

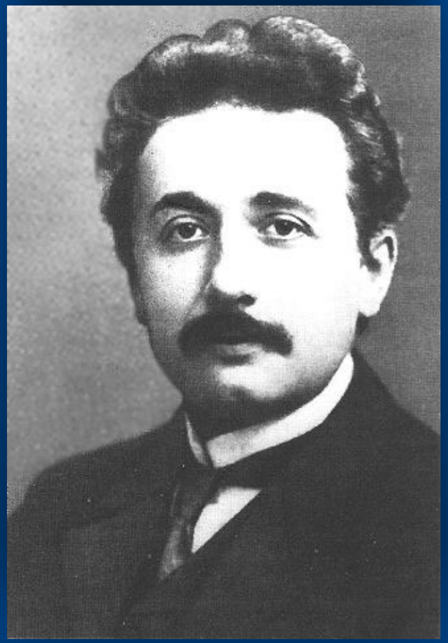


Characteristic Strain
Frequency / Hz
Stochastic background
EPTA
IPTA
eLISA
LISA
10^9 solar mass binaries
10^6 solar mass binaries
Resolvable galactic binaries
Extreme mass ratio inspirals
LIGO
aLIGO
Compact binary inspirals
Supernovae
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
10^{6}

Alternative theories of gravitation

- Scalar-tensor gravity (Brans & Dicke 1961)
- Gravity with extra-dimensions
- f(R) gravity





Scalar-tensor gravity

The masses of the different fundamental particles would not be basic intrinsic properties but a relational property originated in the interaction with some cosmic field.

Brans and Dicke introduced a scalar field that determines the strength of G, i.e. the scalar field determines the coupling strength of matter to gravity.

$$m_i(x^\mu) = \lambda_i \phi(x^\mu).$$

$$\langle \phi \rangle = \frac{1}{G}.$$

$$\Box^2 \phi = 4\pi\lambda (T^{\mathrm{M}})^\mu_\mu,$$

Scalar-tensor gravity

$$S = \int d^4x\sqrt{-g}\left(\frac{\phi R - \omega\frac{\partial_a\phi\partial^a\phi}{\phi}}{16\pi} + \mathcal{L}_{\mathrm{M}}\right)$$

$$G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial_c\phi\partial^c\phi) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)$$

$$\Box\phi = \frac{8\pi}{3+2\omega}T$$

Evidence – derived from the Cassini–Huygens experiment – shows that the value of *w* must exceed 40,000.

In **STVG theory**, gravity is not only an interaction mediated by a tensor field, but has also scalar and vector aspects. The action of the full gravitational field is:

$$S = S_{\mathrm{GR}} + S_{\phi} + S_{\mathrm{S}} + S_{\mathrm{M}},$$

$$S_{\mathrm{GR}} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \frac{1}{G} R,$$

$$S_{\phi} = \omega \int d^4x \sqrt{-g} \left(\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{2} m^2 \phi^{\mu} \phi_{\mu} \right),$$

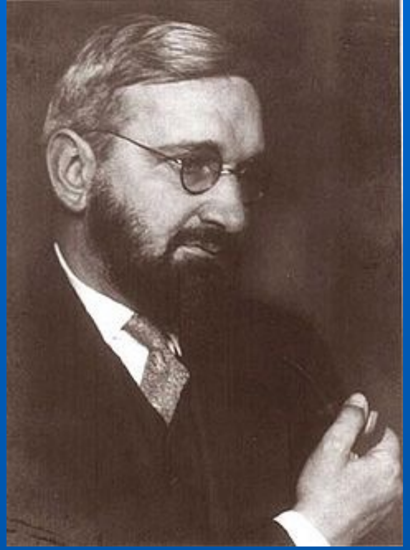
$$S_{\mathrm{S}} = \int d^4x \sqrt{-g} \left[\frac{1}{G^3} \left(\frac{1}{2} g^{\mu\nu} \nabla_{\mu} G \nabla_{\nu} G - V(G) \right) + \frac{1}{Gm^2} \left(\frac{1}{2} g^{\mu\nu} \nabla_{\mu} m \nabla_{\nu} m - V(m) \right) \right].$$

$$B_{\mu\nu} = \partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu}$$

$$G_{\mu\nu} = 8\pi G \left(T^{\mathrm{M}}_{\mu\nu} + T^{\phi}_{\mu\nu} \right),$$

$$\nabla_\nu B^{\nu\mu} = \frac{1}{\omega} J^\mu_{\mathrm{Q}},$$

$$J^\mu_{\mathrm{Q}} = -\frac{1}{\sqrt{-g}} \frac{\delta S_{\mathrm{M}}}{\delta \phi_\mu} = \sqrt{\alpha G_{\mathrm{N}}} J^\mu_{\mathrm{M}},$$



Gravity with extra-dimensions

In April 1919 Kaluza noticed that when he solved Albert Einstein's equations for general relativity using five dimensions, then Maxwell's equations for electromagnetism emerged spontaneously.

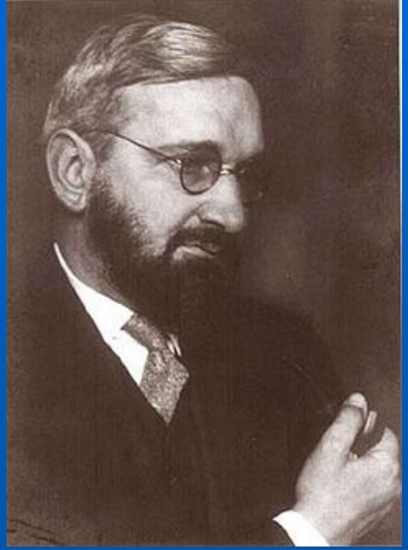
Kaluza's fundamental insight was to write the action as:

$$S = \frac{1}{16\pi\hat{G}} \int_{\mathcal{R}} \hat{R}\sqrt{-\hat{g}}\, d^4x dy,$$

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu,$$

$$x^4 = y$$

$$\frac{\partial \hat{g}_{\mu\nu}}{\partial y} = 0.$$

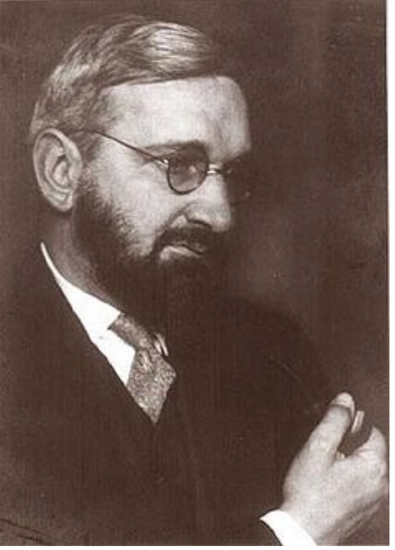


Gravity with extra-dimensions

The five-dimensional metric has 15 components. Ten components are identified with the four-dimensional spacetime metric, four components with the electromagnetic vector potential, and one component with an unidentified scalar field sometimes called the "dilaton".

The five-dimensional Einstein equations yield the four-dimensional Einstein field equations, the Maxwell equations for the electromagnetic field, and an equation for the scalar field. Kaluza also introduced the hypothesis known as the "cylinder condition", that no component of the five-dimensional metric depends on the fifth dimension.

Kaluza also set the scalar field equal to a constant, in which case standard general relativity and electrodynamics are recovered identically.



Gravity with extra-dimensions

$$\tilde{g}_{ab} \equiv \begin{bmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{bmatrix}.$$

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} + \phi^2 A_\mu A_\nu, \qquad \tilde{g}_{5\nu} \equiv \tilde{g}_{\nu 5} \equiv \phi^2 A_\nu, \qquad \tilde{g}_{55} \equiv \phi^2$$

$$ds^2 \equiv \tilde{g}_{ab} dx^a dx^b = g_{\mu\nu} dx^\mu dx^\nu + \phi^2 (A_\nu dx^\nu + dx^5)^2$$

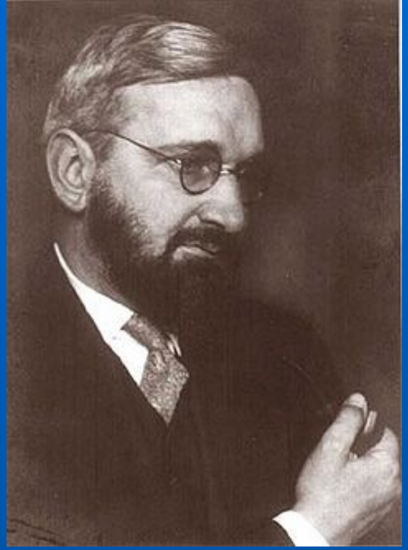
Cylinder condition: $\dfrac{\partial \tilde{g}_{ab}}{\partial x^5} = 0$

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$$

$$\widetilde{R}_{5\alpha} = 0 = \frac{1}{2} g^{\beta\mu} \nabla_\mu (\phi^3 F_{\alpha\beta}) \qquad \widetilde{R}_{55} = 0 \Rightarrow \Box\phi = \frac{1}{4}\phi^3 F^{\alpha\beta} F_{\alpha\beta}$$

$$\widetilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\widetilde{R} = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2}\phi^2 \left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box\phi)$$

This equation shows the remarkable result, called the "Kaluza miracle", that the precise form for the electromagnetic stress-energy tensor emerges from the 5D vacuum equations as a source in the 4D equations: field from the vacuum.



Gravity with extra-dimensions

A very interesting feature of the theory is that charge conservation can be interpreted as momentum conservation in the fifth dimension:

$$J^{\mu} = 2\alpha T^{\mu 5},$$

where J^{μ} is the current density and α a constant. The variation of the action yields both Einstein's and Maxwell's equations:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad \text{and} \quad \partial_{\mu} F^{\mu\nu} = \frac{c^2 \kappa}{2G} J^{\nu}.$$

Gravity with extra-dimensions

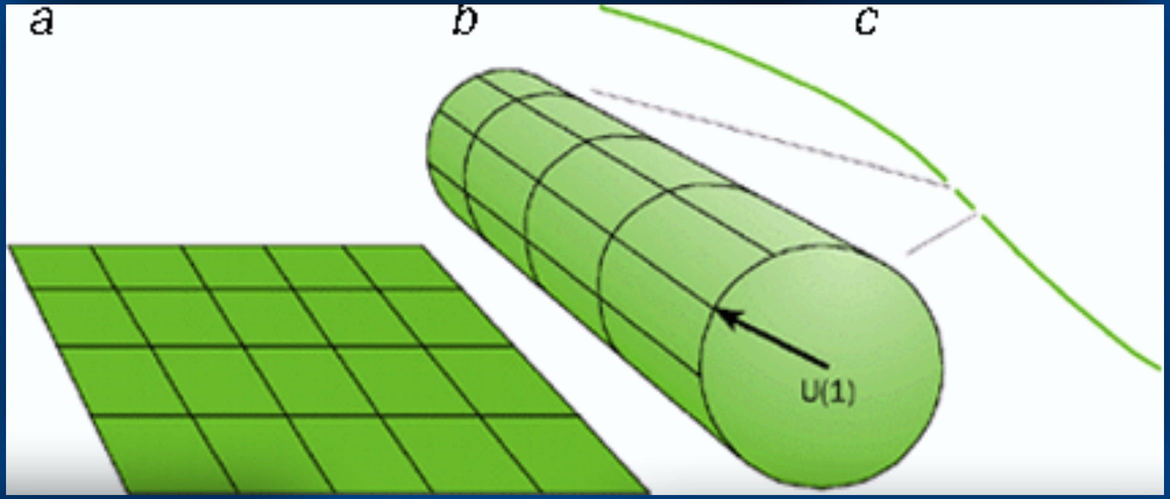
The action introduced by Kaluza describes 4-D gravity along with electromagnetism. The price paid for this unification was the introduction of a scalar field called the dilaton (which was fixed by to be =1) and an extra fifth dimension which is not observed.

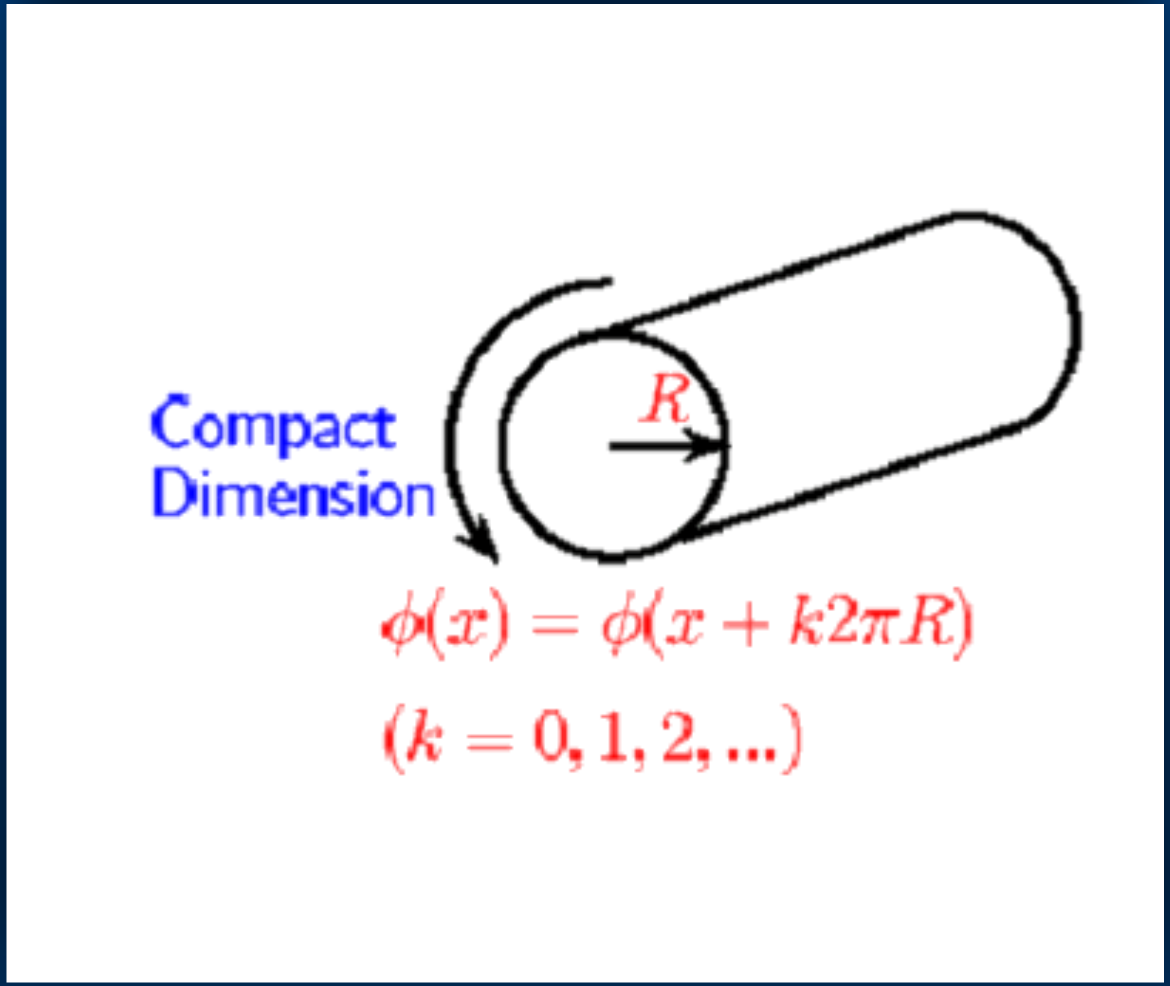
In 1926, Oskar Klein proposed that the fourth spatial dimension is curled up in a circle of very small radius, so that a particle moving a short distance along that axis would return to where it began. The distance a particle can travel before reaching its initial position is said to be the size of the dimension. This extra dimension is a compact set, and the phenomenon of having a space-time with compact dimensions is referred to as compactification.

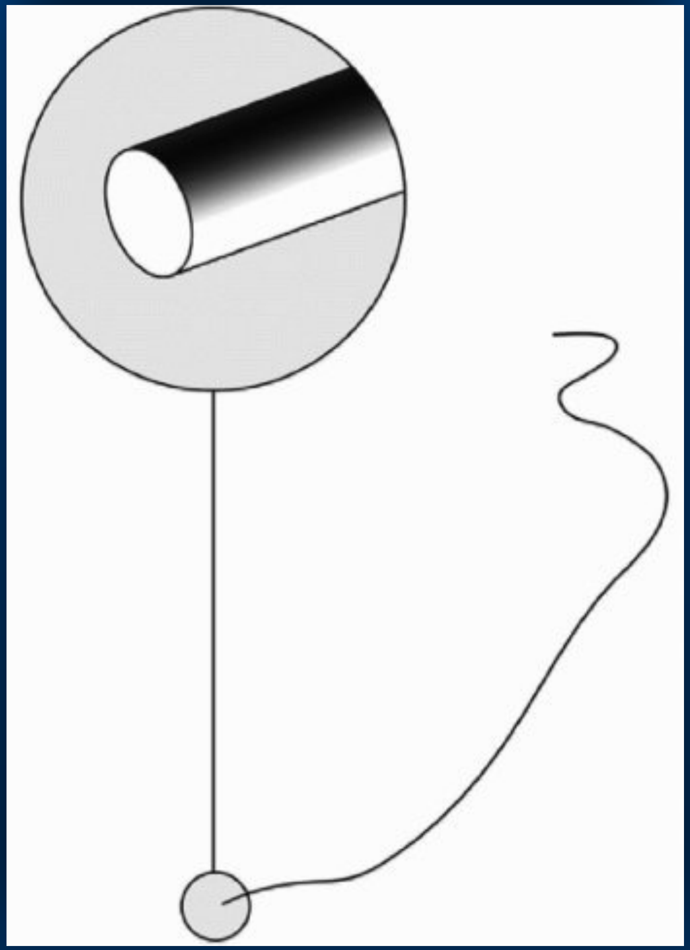
Gravity with extra-dimensions

Klein (1926) suggested that the fifth dimension was not observable because it is *compactified* on a circle. This compactification can be achieved identifying y with $y + 2\pi R$. The quantity R is the size of the extra dimension. Such a size should be extremely small in order to be not detected in experiments. The only natural length of the theory is the Planck length: $R \approx l_{\rm P} \sim 10^{-35}$ m.

Gravity with extra-dimensions







f(R)-Gravity

In f(R) gravity, the Lagrangian of the Einstein-Hilbert action:

$$S[g] = \int \frac{1}{2\kappa} R\sqrt{-g}\, \mathrm{d}^4x$$

is generalized to

$$S[g] = \int \frac{1}{2\kappa} f(R)\sqrt{-g}\, \mathrm{d}^4x,$$

$$f(R) = aR^2 + b\,R$$

$$f(R) = a \exp^{D(R)} + b\,R$$

f(R)-Gravity

$$F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + [g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]F(R) = \kappa T_{\mu\nu},$$

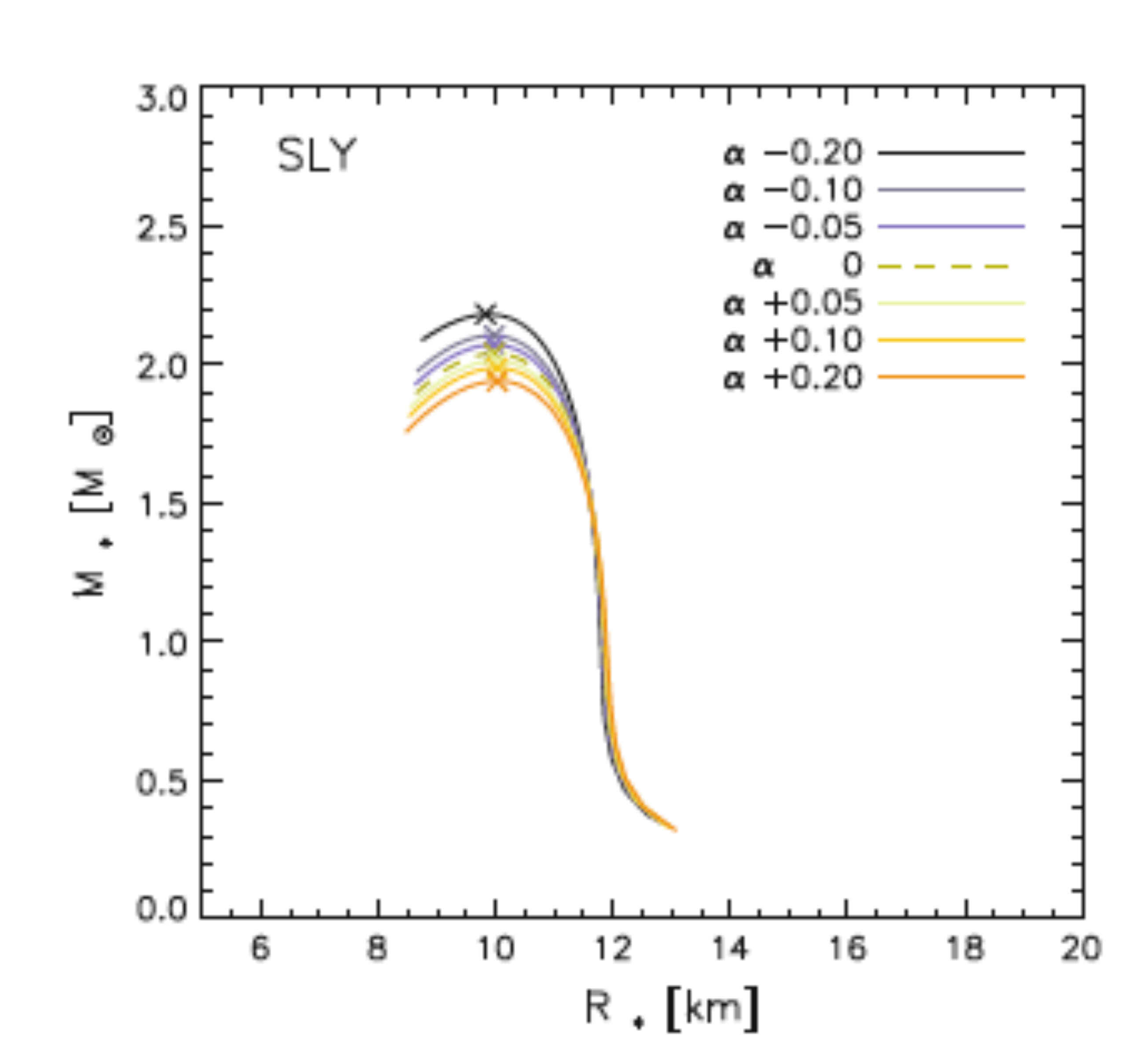
$$F(R) = \frac{\partial f(R)}{\partial R}.$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,L_{\mathrm{m}})}{\delta g^{\mu\nu}},$$

Higher than second order derivatives are possible in f(R) theory depending on the explicit form of the function f

$$S = \frac{c^4}{16\pi G} \int \mathrm{d}^4x \sqrt{-g}(R + \alpha R^2) + S_{\mathrm{matter}},$$

$$G_{\mu\nu} + \alpha\left[2R\left(R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu}\right) + 2\left(g_{\mu\nu}\Box R - \nabla_\mu\nabla_\nu R\right)\right] = \frac{8\pi G}{c^4}T_{\mu\nu},$$



Maxwell equations

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}.$$

$$[\,j^\mu] = \rho_0 \gamma_u (c, \vec{u}) = (c\rho, \vec{j}),$$

Maxwell equations

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu,$$

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0.$$

$$\partial_\mu j^\mu = 0.$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0,$$

+

$$m_0 \frac{du^\mu}{d\tau} = qF^\mu{}_\nu u^\nu.$$

Maxwell equations

$$[A^\mu] = \left(\frac{\phi}{c}, \vec{A}\right),$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

$$\Box^2 A_\mu = \mu_0 j_\mu,$$

Maxwell equations with gravity

$$\nabla_\mu F^{\mu\nu} = \mu_0 j^\nu ,$$

$$\nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0.$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{q}{m_0} F^\mu{}_\nu \frac{dx^\nu}{d\tau} .$$

Born-Infeld non-linear electrodynamics

$$L_{\mathrm{BI}} = \frac{1}{4\pi b^2}\left(1 - \sqrt{1 + \frac{1}{2}F_{\sigma\nu}F^{\sigma\nu}b^2 - \frac{1}{4}\tilde{F}_{\sigma\nu}F^{\sigma\nu}b^4}\right),$$

$$L_{\mathrm{BI}} = \frac{b^2}{4\pi}\left[1 - \sqrt{1 - \frac{B^2 - E^2}{b^2} - \frac{(\mathbf{E}\cdot\mathbf{B})^2}{b^4}}\right].$$

Einstein-Maxwell equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + E_{\mu\nu}),$$

$$\frac{4\pi}{c} E^{\mu}_{\nu} = -F^{\mu\rho} F_{\rho\nu} + \frac{1}{4}\delta^{\mu}_{\nu} F^{\sigma\lambda} F_{\sigma\lambda},$$

$$F_{\mu\nu} = A_{\mu;\nu} - A_{\nu;\mu},$$

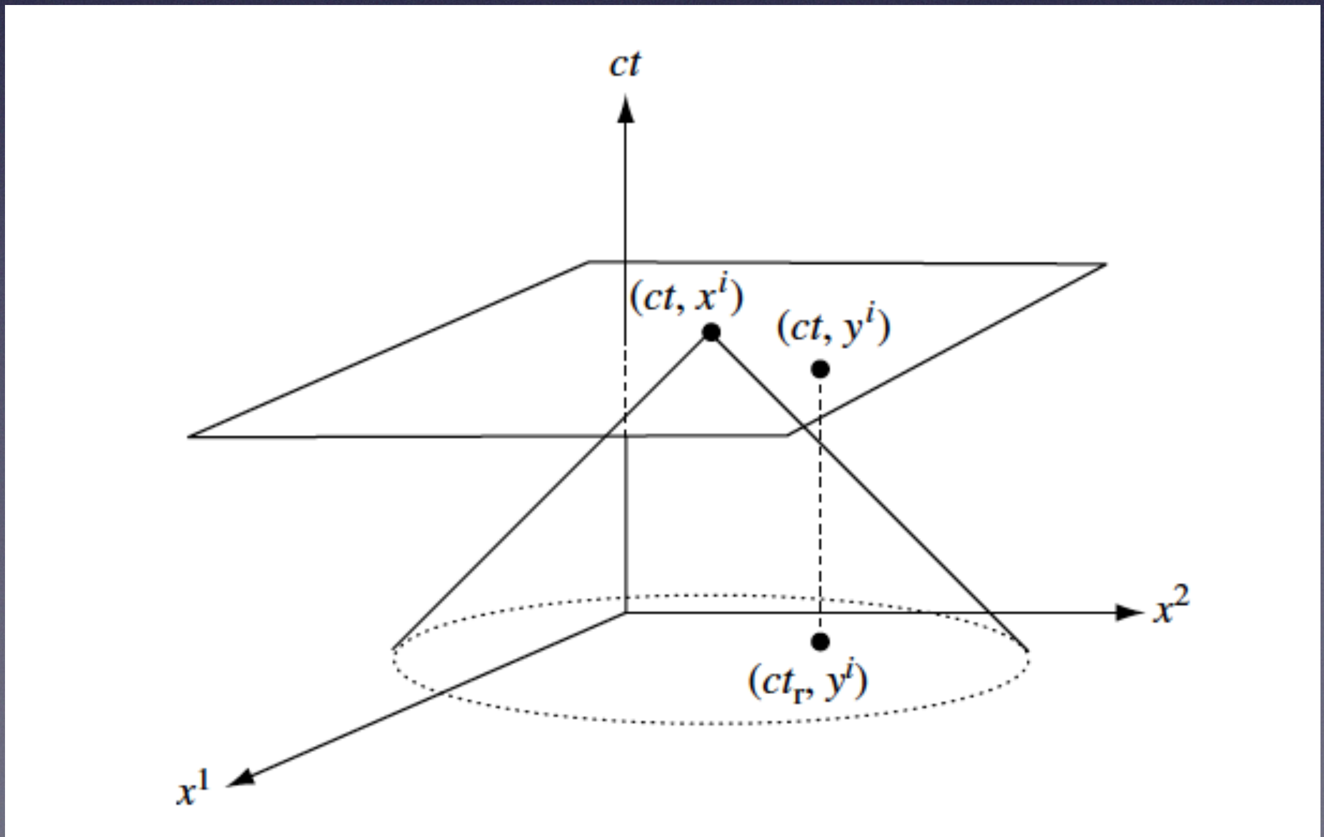
$$F^{\nu;\nu}_{\mu} = \frac{4\pi}{c} J_{\mu}.$$

Energy of gravitational waves

$$\Box^2 \bar{h}^{\mu\nu} = -2\kappa T^{\mu\nu},$$

$$\partial_\mu \bar{h}^{\mu\nu} = 0.$$

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4} \int \frac{T^{\mu\nu}(ct - |\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \, d^3\vec{y}.$$



$$\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{r} + (-y^i)\partial_i\left(\frac{1}{r}\right) + \frac{1}{2!}(-y^i)(-y^j)\partial_i\partial_j\left(\frac{1}{r}\right) + \cdots,$$

$$= \frac{1}{r} + y^i\,\frac{x_i}{r^3} + y^i y^j\left(\frac{3x_i x_j - r^2\delta_{ij}}{r^5}\right) + \cdots,$$

$$\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4}\left[\frac{1}{r}\int T^{\mu\nu}(ct_{\mathrm{r}},\vec{y})\,d^3\vec{y} + \frac{x_i}{r^3}\int T^{\mu\nu}(ct_{\mathrm{r}},\vec{y})y^i\,d^3\vec{y} \right.$$
$$\left. + \frac{3x_i x_j - r^2\delta_{ij}}{r^5}\int T^{\mu\nu}(ct_{\mathrm{r}},\vec{y})y^i y^j\,d^3\vec{y} + \cdots\right],$$

Multipolar expansion

$$\bar{h}^{\mu\nu}(ct,\vec{x}) = -\frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}M^{\mu\nu i_1 i_2\cdots i_\ell}(ct_{\mathrm{r}})\partial_{i_1}\partial_{i_2}\cdots\partial_{i_\ell}\left(\frac{1}{r}\right),$$

$$M^{\mu\nu i_1 i_2\cdots i_\ell}(ct) = \int T^{\mu\nu}(ct,\vec{y})y^{i_1}y^{i_2}\cdots y^{i_\ell}\,d^3\vec{y}.$$

Compact source approximation

$$\bar{h}^{\mu\nu}(ct, \vec{x}) = -\frac{4G}{c^4 r} \int T^{\mu\nu}(ct - r, \vec{y})\, d^3\vec{y}.$$

$\int T^{00}\, d^3\vec{y}$, total energy of source particles (including rest mass energy) $\equiv Mc^2$;
$\int T^{0i}\, d^3\vec{y}$, $c\times$ total momentum of source particles in the x^i-direction $\equiv P^i c$;
$\int T^{ij}\, d^3\vec{y}$, integrated internal stresses in the source.

$$\partial_0 T^{00} + \partial_k T^{0k} = 0,$$
$$\partial_0 T^{i0} + \partial_k T^{ik} = 0.$$

$$\bar{h}^{00} = -\frac{4GM}{c^2 r}, \qquad \bar{h}^{i0} = \bar{h}^{0i} = 0.$$

$$\bar{h}^{ij}(ct, \vec{x}) = -\frac{2G}{c^6 r}\left[\frac{d^2 I^{ij}(ct')}{dt'^2}\right]_{\mathrm{r}},$$

$$I^{ij}(ct) = \int T^{00}(ct, \vec{y})\, y^i y^j\, d^3\vec{y},$$

Quadrupole-moment tensor of the energy density of the source

$$G^{(1)}_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + t_{\mu\nu}),$$

$$G^{(1)}_{\mu\nu} + \frac{8\pi G}{c^4} t_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}.$$

$$G_{\mu\nu} \equiv G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = -\frac{8\pi G}{c^4} T_{\mu\nu},$$

This suggests that, to a good approximation, we should make the identification:

$$t_{\mu\nu} \equiv \frac{c^4}{8\pi G} G^{(2)}_{\mu\nu}.$$

$$G^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)} - \frac{1}{2}h_{\mu\nu}R^{(1)} + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma},$$

$$t_{\mu\nu} \equiv \frac{c^4}{8\pi G}\left\langle G^{(2)}_{\mu\nu}\right\rangle.$$

$$R^{(2)}_{\mu\nu} = \partial_\nu {\Gamma^{(2)}}^\sigma{}_{\mu\sigma} - \partial_\sigma {\Gamma^{(2)}}^\sigma{}_{\mu\nu} + {\Gamma^{(1)}}^\rho{}_{\mu\sigma}{\Gamma^{(1)}}^\sigma{}_{\rho\nu} - {\Gamma^{(1)}}^\rho{}_{\mu\nu}{\Gamma^{(1)}}^\sigma{}_{\rho\sigma},$$

$$\begin{aligned}\Gamma^\sigma{}_{\mu\nu} &= {\Gamma^{(1)}}^\sigma{}_{\mu\nu} + {\Gamma^{(2)}}^\sigma{}_{\mu\nu} + \cdots \\ &= \frac{1}{2}(\partial_\nu h^\sigma_\mu + \partial_\mu h^\sigma_\nu - \partial^\sigma h_{\mu\nu}) - \frac{1}{2}h^{\sigma\tau}(\partial_\nu h_{\tau\mu} + \partial_\mu h_{\tau\nu} - \partial_\tau h_{\mu\nu}) + \cdots .\end{aligned}$$

$$R^{(2)}_{\mu\nu} = -\tfrac{1}{4}(\partial_\mu h^{\rho\sigma})\partial_\nu h_{\rho\sigma} + \tfrac{1}{2}h^{\rho\sigma}(\partial_\mu\partial_\sigma h_{\nu\rho} + \partial_\nu\partial_\sigma h_{\mu\rho} - \partial_\mu\partial_\nu h_{\rho\sigma} - \partial_\rho\partial_\sigma h_{\mu\nu})$$
$$+ \tfrac{1}{2}(\partial^\sigma h^\rho_\nu)(\partial_\rho h_{\sigma\mu} - \partial_\sigma h_{\rho\mu}) + \tfrac{1}{2}(\partial_\sigma h^{\rho\sigma} - \tfrac{1}{2}\partial^\rho h)(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho} - \partial_\rho h_{\mu\nu}).$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle(\partial_\mu \bar{h}_{\rho\sigma})\partial_\nu \bar{h}^{\rho\sigma} - 2(\partial_\sigma \bar{h}^{\rho\sigma})\partial_{(\mu}\bar{h}_{\nu)\rho} - \frac{1}{2}(\partial_\mu \bar{h})\partial_\nu \bar{h}\right\rangle,$$

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle\left(\partial_\mu h^{\mathrm{TT}}_{\rho\sigma}\right)\partial_\nu h^{\rho\sigma}_{\mathrm{TT}}\right\rangle.$$

$$A^{00}_{\mathrm{TT}} = 0 \qquad \text{and} \qquad A^{ij}_{\mathrm{TT}}k_j = 0.$$

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \left(\partial_\mu h_{ij}^{\mathrm{TT}}\right) \partial_\nu h_{\mathrm{TT}}^{ij} \right\rangle.$$

$$F(\vec{n}) = -ct^{0k} n_k,$$

$$h_{\mathrm{TT}}^{ij} = A_{\mathrm{TT}}^{ij} \cos k_\lambda x^\lambda,$$

$$\langle \sin^2(k_\lambda x^\lambda) \rangle = \tfrac{1}{2}$$

$$t_{\mu\nu} = \frac{c^4}{64\pi G} k_\mu k_\nu A^{ij}_{\mathrm{TT}} A^{\mathrm{TT}}_{ij}.$$

$$k^0 = |\vec{k}| = -k^l \hat{k}_l,$$

$$F = -ct^{0l}\hat{k}_l = -\frac{c^5}{64\pi G} k^0 k^l \hat{k}_l A^{ij}_{\mathrm{TT}} A^{\mathrm{TT}}_{ij} = \frac{c^5}{64\pi G} k^0 k^0 A^{ij}_{\mathrm{TT}} A^{\mathrm{TT}}_{ij} = ct^{00}$$

The final expression is simply the energy density associated with the plane wave multiplied by its speed, and hence makes good physical sense as the energy flux carried by the wave in its direction of propagation.

$$\frac{dE}{dt} = -L_{\mathrm{GW}} = -r^2 \int_{4\pi} F(\vec{e}_r)\, d\Omega,$$

$$F(\vec{n}) = -\frac{c^4}{32\pi G}\left\langle \left(\partial_t h_{ij}^{\mathrm{TT}}\right)\left(\partial_k h^{ij}_{\mathrm{TT}}\right)\right\rangle n^k$$

$$\bar{h}^{ij} = -\frac{2G}{c^6 r}\left[\ddot{I}^{ij}\right]_{\mathrm{r}},$$

$$J_{ij} = I_{ij} - \tfrac{1}{3}\delta_{ij} I,$$

$$F(\vec{e}_r) = \frac{G}{8\pi r^2 c^9} \left\langle \left[\dddot{J}_{ij}^{\mathrm{TT}} \dddot{J}^{ij}_{\mathrm{TT}} \right]_{\mathrm{r}} \right\rangle .$$

$$\frac{dE}{dt} = -L_{\mathrm{GW}} = -\frac{G}{5c^9} \left\langle \left[\dddot{J}_{ij} \dddot{J}^{ij} \right]_{\mathrm{r}} \right\rangle .$$